Nonlinear Control
Lecture 9: Feedback Linearization

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Feedback Linearization

- **The main idea is:** algebraically transform a nonlinear system dynamics into a (fully or partly) linear one, so that linear control techniques can be applied.

- In its simplest form, feedback linearization cancels the nonlinearities in a nonlinear system so that the closed-loop dynamics is in a linear form.

- **Example:** Controlling the fluid level in a tank
  - **Objective:** controlling of the level $h$ of fluid in a tank to a specified level $h_d$, using control input $u$
  - the initial level is $h_0$. 

![Fluid level control in a tank](image)
Example Cont’d

The dynamics:

\[ A(h) \dot{h}(t) = u - a \sqrt{2gh} \]

where \( A(h) \) is the cross section of the tank and \( a \) is the cross section of the outlet pipe.

Choose \( u = a \sqrt{2gh} + A(h) \dot{v} \Rightarrow \dot{h} = v \)

Choose the equivalent input \( v: \; v = -\alpha \tilde{h} \) where \( \tilde{h} = h(t) - h_d \) is error level, \( \alpha \) a pos. const.

\[ : \text{resulting closed-loop dynamics: } \dot{h} + \alpha \tilde{h} = 0 \Rightarrow \tilde{h} \rightarrow 0 \text{ as } t \rightarrow \infty \]

The actual input flow: \( u = a \sqrt{2gh} + A(h) \alpha \tilde{h} \)

\[ : \text{First term provides output flow } a \sqrt{2gh} \]

\[ : \text{Second term raises the fluid level according to the desired linear dynamics} \]

If \( h_d \) is time-varying: \( v = \dot{h}_d(t) - \alpha \tilde{h} \)

\[ : \tilde{h} \rightarrow 0 \text{ as } t \rightarrow \infty \]
Canceling the nonlinearities and imposing a desired linear dynamics, can be simply applied to a class of nonlinear systems, so-called companion form, or controllability canonical form:

A system in companion form:

\[
x^{(n)}(t) = f(x) - b(x)u
\]  

- \( u \) is the scalar control input
- \( x \) is the scalar output; \( x = [x, \dot{x}, ..., x^{(n-1)}] \) is the state vector.
- \( f(x) \) and \( b(x) \) are nonlinear functions of the states.
- no derivative of input \( u \) presents.

\( (1) \) can be presented as controllability canonical form

\[
\frac{d}{dt} \begin{bmatrix} x_1 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} x_2 \\ \vdots \\ x_n \end{bmatrix} f(x) + b(x)u
\]

for nonzero \( b \), define control input: \( u = \frac{1}{b}[v - f] \)
Feedback Linearization

- the control law:

\[ v = -k_0x - k_1\dot{x} - \ldots - k_{n-1}x^{(n-1)} \]

- \( k_i \) is chosen s.t. the roots of \( s^n + k_{n-1}s^{n-1} + \ldots + k_0 \) are strictly in LHP.

- **Thus**: \( x^{(n)} + k_{n-1}x^{(n-1)} + \ldots + k_0 = 0 \) is e.s.

- For tracking desired output \( x_d \), the control law is:

\[ v = x_d^{(n)} - k_0x - k_1\dot{x} - \ldots - k_{n-1}x^{(n-1)} \]

- Exponentially convergent tracking, \( e = x - x_d \to 0 \).

- This method is extendable when the scalar \( x \) was replaced by a vector and the scalar \( b \) by an invertible square matrix.

- When \( u \) is replaced by an invertible function \( g(u) \mapsto u = g^{-1}(\frac{1}{b}[v - f]) \),
Example: Feedback Linearization of a Two-link Robot

A two-link robot: each joint equipped with
  - a motor for providing input torque
  - an encoder for measuring joint position
  - a tachometer for measuring joint velocity

objective: the joint positions $q_1$ and $q_2$ follow desired position histories $q_{dl}(t)$ and $q_{d2}(t)$

For example when a robot manipulator is required to move along a specified path, e.g., to draw circles.
Using the Lagrangian equations the robotic dynamics are:

\[
\begin{bmatrix}
H_{11} & H_{12} \\
H_{21} & H_{22}
\end{bmatrix}
\begin{bmatrix}
\ddot{q}_1 \\
\ddot{q}_2
\end{bmatrix}
+ \begin{bmatrix}
-h\dot{q}_2 & -h\dot{q}_2 - h\dot{q}_1 \\
h\dot{q}_1 & 0
\end{bmatrix}
\begin{bmatrix}
\dot{q}_1 \\
\dot{q}_2
\end{bmatrix}
+ \begin{bmatrix}
g_1 \\
g_2
\end{bmatrix}
= \begin{bmatrix}
\tau_1 \\
\tau_2
\end{bmatrix}
\]

where \( q = [q_1 \ q_2]^T \): the two joint angles, \( \tau = [\tau_1 \ \tau_2]^T \): the joint inputs, and

\[
H_{11} = m_1 l_{c1}^2 + l_1 + m_2[l_1^2 + l_{c2}^2 + 2l_1 l_{c2} \cos q_2] + l_2
\]

\[
H_{22} = m_2 l_{c2}^2 + l_2 H_{12} = H_{21} = m_2 l_1 l_{c2} \cos q_2 + m_2 l_{c2}^2 + l_2
\]

\[
g_1 = m_1 l_{c1} \cos q_1 + m_2 g[l_{c2} \cos (q_1 + q_2) + l_1 \cos q_1]
\]

\[
g_2 = m_2 l_{c2} g \cos (q_1 + q_2), \ h = m_2 l_1 l_{c2} \sin q_2
\]

Control law for tracking, (computed torque):

\[
\begin{bmatrix}
\tau_1 \\
\tau_2
\end{bmatrix}
= \begin{bmatrix}
H_{11} & H_{12} \\
H_{21} & H_{22}
\end{bmatrix}
\begin{bmatrix}
v_1 \\
v_2
\end{bmatrix}
+ \begin{bmatrix}
-h\dot{q}_2 & -h\dot{q}_2 - h\dot{q}_1 \\
h\dot{q}_1 & 0
\end{bmatrix}
\begin{bmatrix}
\dot{q}_1 \\
\dot{q}_2
\end{bmatrix}
+ \begin{bmatrix}
g_1 \\
g_2
\end{bmatrix}
\]

where \( v = \dddot{q}_d - 2\lambda \ddot{q} - \lambda^2 \dot{q}, \ \dddot{q} = q - q_d \): position tracking error, \( \lambda \): pos. const.

\[
\dddot{q}_d + 2\lambda \dddot{q} + \lambda^2 \dot{q} = 0
\]

This method is applicable for arbitrary # of links.
Input-State Linearization

- When the nonlinear dynamics is not in a controllability canonical form, use algebraic transformations.

- Consider the SISO system

\[ \dot{x} = f(x, u) \]

- In input-state linearization technique:
  1. finds a state transformation \( z = z(x) \) and an input transformation \( u = u(x, v) \) s.t. the nonlinear system dynamics is transformed into \( \dot{z} = Az + bv \)
  2. Use standard linear techniques (such as pole placement) to design \( v \).
Example:

Consider

\[
\begin{align*}
\dot{x}_1 &= -2x_1 + ax_2 + \sin x_1 \\
\dot{x}_2 &= -x_2 \cos x_1 + u \cos(2x_1)
\end{align*}
\]

- Equ. pt. (0, 0)
- The nonlinearity cannot be directly canceled by the control input \( u \)
- Define a new set of variables:

\[
\begin{align*}
z_1 &= x_1 \\
z_2 &= ax_2 + \sin x_1 \\
\dot{z}_1 &= -2z_1 + z_2 \\
\dot{z}_2 &= -2z_1 \cos z_1 + \cos z_1 \sin z_1 + au \cos(2z_1)
\end{align*}
\]

- The Equ. pt. is still (0, 0).
- The control law: \( u = \frac{1}{a \cos(2z_1)} (v - \cos z_1 \sin z_1 + 2z_1 \cos z_1) \)
- The new dynamics is linear and controllable: \( \dot{z}_1 = -2z_1 + z_2, \quad \dot{z}_2 = v \)
- By proper choice of feedback gains \( k_1 \) and \( k_2 \) in \( v = -k_1 z_1 - k_2 z_2 \), place the poles properly.
- Both \( z_1 \) and \( z_2 \) converge to zero, \( \rightarrow \) the original state \( x \) converges to zero.
The result is not global.

- The result is not valid when \( x_l = (\pi/4 \pm k\pi/2), \ k = 0, 1, 2, ... \)

The input-state linearization is achieved by a combination of a state transformation and an input transformation with state feedback used in both.

To implement the control law, the new states \((z_1, z_2)\) must be available.

- If they are not physically meaningful or measurable, they should be computed by measurable original state \(x\).

If there is uncertainty in the model, e.g., on \(a\rightarrow\) error in the computation of new state \(z\) as well as control input \(u\).

For tracking control, the desired motion needs to be expressed in terms of the new state vector.

Two questions arise for more generalizations which will be answered in next lectures:

- What classes of nonlinear systems can be transformed into linear systems?
- How to find the proper transformations for those which can?
Input-Output Linearization

Consider

\[
\dot{x} = f(x, u) \\
y = h(x)
\]

Objective: tracking a desired trajectory \(y_d(t)\), while keeping the whole state bounded.

\(y_d(t)\) and its time derivatives up to a sufficiently high order are known and bounded.

The difficulty: output \(y\) is only indirectly related to the input \(u\).

\[
\therefore \text{it is not easy to see how the input } u \text{ can be designed to control the tracking behavior of the output } y.
\]

Input-output linearization approach:

1. Generating a linear input-output relation
2. Formulating a controller based on linear control
Example:

- Consider

\[
\begin{align*}
\dot{x}_1 &= \sin x_2 + (x_2 + 1)x_3 \\
\dot{x}_2 &= x_1^5 + x_3 \\
\dot{x}_3 &= x_1^2 + u \\
y &= x_1
\end{align*}
\]

- To generate a direct relationship between the output \( y \) and the input \( u \), differentiate the output \( \dot{y} = \dot{x}_1 = \sin x_2 + (x_2 + 1)x_3 \)

- No direct relationship \( \implies \) differentiate again: \( \ddot{y} = (x_2 + 1)u + f(x) \), where 
  \[
  f(x) = (x_1^5 + x_3)(x_3 + \cos x_2) + (x_2 + 1)x_1^2
  \]

- Control input law: \( u = \frac{1}{x_2+1}(v - f) \).

- Choose \( v = \ddot{y}_d - k_1 e - k_2 \dot{e} \), where \( e = y - y_d \) is tracking error, \( k_1 \) and \( k_2 \) are pos. const.

- The closed-loop system: \( \dot{e} + k_2 \dot{e} + k_1 e = 0 \)

\[ \therefore \text{e.s. of tracking error} \]
Example Cont’d

- The control law is defined everywhere except at singularity points s.t. \( x_2 = -1 \)
- To implement the control law, full state measurement is necessary, because the computations of both the derivative \( y \) and the input transformation need the value of \( x \).
- If the output of a system should be differentiated \( r \) times to generate an explicit relation between \( y \) and \( u \), the system is said to have relative degree \( r \).
  - For linear systems this terminology expressed as \( \# \) poles − \( \# \) zeros.
- For any controllable system of order \( n \), by taking at most \( n \) differentiations, the control input will appear to any output, i.e., \( r \leq n \).
  - If the control input never appears after more than \( n \) differentiations, the system would not be controllable.
Feedback Linearization

**Internal dynamics**: a part of dynamics which is unobservable in the input-output linearization.

- In the example it can be \( \dot{x}_3 = x_1^2 + \frac{1}{x_2+1}(\dot{y}_d(t) - k_1e - k_2 \dot{e} + f) \)

- The desired performance of the control based on the reduced-order model depends on the stability of the internal dynamics.
  - stability in BIBO sense

**Example**: Consider

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} =
\begin{bmatrix}
x_3^3 + u \\
x_2 \\
y = x_1
\end{bmatrix}
\]  

(2)

- Control objective: \( y \) tracks \( y_d \).
  - First differentiations of \( y \) \( \rightsquigarrow \) linear I-O relation
  - The control law \( u = -x_2^3 - e(t) - \dot{y}_d(t) \ \rightsquigarrow \) exp. convergence of \( e \):
    \( \dot{e} + e = 0 \)
  - Internal dynamics: \( \dot{x}_2 + x_2^3 = \dot{y}_d - e \)
  - Since \( e \) and \( \dot{y}_d \) are bounded (\( \dot{y}_d(t) - e \leq D \)), \( x_2 \) is ultimately bounded.
I-O linearization can also be applied to stabilization \((y_d(t) \equiv 0)\):

- For previous example the objective will be \(y\) and \(\dot{y}\) will be driven to zero and stable internal dynamics guarantee stability of the whole system.
- No restriction to choose physically meaningful \(h(x)\) in \(y = h(x)\)
- Different choices of output function leads to different internal dynamics which some of them may be unstable.

When the relative degree of a system is the same as its order:

- There is no internal dynamics
- The problem will be input-state linearization
Summary

- Feedback linearization cancels the nonlinearities in a nonlinear system s.t. the closed-loop dynamics is in a linear form.

- Canceling the nonlinearities and imposing a desired linear dynamics, can be applied to a class of nonlinear systems, named companion form, or controllability canonical form.

- When the nonlinear dynamics is not in a controllability canonical form, input-state linearization technique is employed:
  1. Transform input and state into companion canonical form
  2. Use standard linear techniques to design controller

- For tracking a desired traj, when $y$ is not directly related to $u$, I-O linearization is applied:
  1. Generating a linear input-output relation (take derivative of $y \leq n$ times)
  2. Formulating a controller based on linear control

- Relative degree: # of differentiating $y$ to find explicit relation to $u$.

- If $r \neq n$, there are $n - r$ internal dynamics that their stability be checked.
Internal Dynamics of Linear Systems

- In general, directly determining the stability of the internal dynamics is not easy since it is nonlinear, nonautonomous, and coupled to the “external” closed-loop dynamics.

- We are seeking to translate the concept of internal dynamics to the more familiar context of linear systems.

**Example:** Consider the controllable, observable system

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = \begin{bmatrix}
x_2 + u \\
u
\end{bmatrix}
\]

\(y = x_1\)  \hspace{1cm} (3)

- Control objective: \(y\) tracks \(y_d\).
  - First differentiations of \(y\)\(\rightarrow\) \(\dot{y} = x_2 + u\)
  - The control law \(u = -x_2 - e(t) - \dot{y}_d(t)\)\(\rightarrow\) exp. convergence of \(e: \dot{e} + e = 0\)
  - Internal dynamics: \(\dot{x}_2 + x_2 = \dot{y}_d - e\)
  - \(e\) and \(\dot{y}_d\) are bounded \(\rightarrow\) \(x_2\) and therefore \(u\) are bounded.
Now consider a little different dynamics

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = \begin{bmatrix}
x_2 + u \\
-u
\end{bmatrix}
\]

\[
y = x_1
\]

Using the same control law yields the following internal dynamics

\[
\dot{x}_2 - x_2 = e(t) - \dot{y}_d
\]

Although \(y_d\) and \(y\) are bounded, \(x_2\) and \(u\) diverge to \(\infty\) as \(t \to \infty\).

Why the same tracking design method yields different results?

- Transfer function of (3) is: \(W_1(s) = \frac{s+1}{s^2}\).
- Transfer function of (4) is: \(W_2(s) = \frac{s-1}{s^2}\).
- Both have the same poles but different zeros.
- The system \(W_1\) which is **minimum-phase** tracks the desired trajectory perfectly.
- The system \(W_2\) which is **nonminimum-phase** requires infinite effort for tracking.
Consider a third-order linear system with one zero

\[ \dot{x} = Ax + bu, \quad y = c^T x \] (5)

Its transfer function is:

\[ y = \frac{b_0 + b_1 s}{a_0 + a_1 s + a_2 s^2 + a_3 s^3} u \]

First transform it into the companion form:

\[
\begin{bmatrix}
\dot{z}_1 \\
\dot{z}_2 \\
\dot{z}_3 \\
\end{bmatrix} = \begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
-a_0 & -a_1 & -a_2 \\
\end{bmatrix} \begin{bmatrix}
z_1 \\
z_2 \\
z_3 \\
\end{bmatrix} + \begin{bmatrix}
0 \\
0 \\
1 \\
\end{bmatrix} u \] (6)

\[ y = \begin{bmatrix} b_0 & b_1 & 0 \end{bmatrix} \begin{bmatrix}
z_1 \\
z_2 \\
z_3 \\
\end{bmatrix} \]
In second derivation of $y$, $u$ appears:
\[
\ddot{y} = b_0 z_3 + b_1 (-a_0 z_1 - a_1 z_2 - a_2 z_3 + u)
\]

\[\therefore\] Required number of differentiations (the relative degree) is indeed the same as \# of poles - \# of zeros

- Note that: the I-O relation is independent of the choice of state variables
- \[\approx\] two differentiations is required for $u$ to appear if we use (5).

The control law:
\[
u = (a_0 z_1 + a_1 z_2 + a_2 z_3 - \frac{b_0}{b_1} z_3) + \frac{1}{b_1} (-k_1 e - k_2 \dot{e} - \ddot{y}_d)
\]

\[\therefore\] an exp. stable tracking is guaranteed

The internal dynamics can be described by only one state equation

- $z_1$ can complete the state vector, ($z_1$, $y$, and $\dot{y}$ are related to $z_1$, $z_2$ and $z_3$ through a one-to-one transformation).
- $\dot{z}_1 = z_2 = \frac{1}{b_1} (y - b_0 z_1)$
- $y$ is bounded $\approx$ stability of the internal dynamics depends on $-\frac{b_0}{b_1}$
- If the system is minimum phase the internal dynamics is stable (independent of initial conditions and magnitude of desired trajectory)
Zero-Dynamics

- For linear systems the stability of the internal dynamics is determined by the locations of the zeros.

- To extend the results for nonlinear systems the concept of zero should be modified.

- Extending the notion of zeros to nonlinear systems is not trivial
  - In linear systems I-O relation is described by transfer function which zeros and poles are its fundamental components. But in nonlinear systems we cannot define transfer function
  - Zeros are intrinsic properties of a linear plant. But for nonlinear systems the stability of the internal dynamics may depend on the specific control input.

- Zero dynamics: is defined to be the internal dynamics of the system when the system output is kept at zero by the input. (output and all of its derivatives)
For dynamics (2), the zero dynamics is $\dot{x}_2 + x_2^3 = 0$

- we find input $u$ to maintain the system output at zero uniquely (keep $x_1$ zero in this example),
- By Layap. Fcn $V = x_2^2$ it can be shown it is a.s

For linear system (5), the zero dynamics is $\dot{z}_1 + (b_0/b_1)z_1 = 0$

∴ The poles of the zero-dynamics are exactly the zeros of the system.

In linear systems, if all zeros are in LHP $\Rightarrow$ g.a.s. of the zero-dynamics $\Rightarrow$ g.s. of internal dynamics.

In nonlinear systems, no results on the global stability
- only local stability is guaranteed for the internal dynamics even if the zero-dynamics is g.e.s.

Zero-dynamics is an intrinsic feature of a nonlinear system, which does not depend on the choice of control law or the desired trajectories.

Examining the stability of zero-dynamics is easier than examining the stability of internal dynamics, But the result is local.
- Zero-dynamics only involves the internal states
- Internal dynamics is coupled to the external dynamics and desired trajs.
Zero-Dynamics

- Similar to the linear case, a nonlinear system whose zero dynamics is asymptotically stable is called an asymptotically minimum phase system.
- If the zero-dynamics is unstable, different control strategies should be sought.
- As summary control design based on input-output linearization is in three steps:
  1. Differentiate the output $y$ until the input $u$ appears
  2. Choose $u$ to cancel the nonlinearities and guarantee tracking convergence
  3. Study the stability of the internal dynamics
- If the relative degree associated with the input-output linearization is the same as the order of the system $\rightarrow$ the nonlinear system is fully linearized $\rightarrow$ satisfactory controller
- Otherwise, the nonlinear system is only partly linearized $\rightarrow$ whether or not the controller is applicable depends on the stability of the internal dynamics.
Preliminary Mathematics

- To formalize and generalize the previous intuitive concepts for a broad class of nonlinear systems, let us introduce some mathematical tools.

- Vector function \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is called a vector field in \( \mathbb{R}^n \).

- Smooth vector field: function \( f(x) \) has continuous partial derivatives of any required order.

- Gradient of a smooth scalar function \( h(x) \) is denoted by
  \[
  \nabla h = \frac{\partial h}{\partial x}, \quad \text{where} \quad (\nabla h)_j = \frac{\partial h}{\partial x_j}
  \]

- Jacobian of a vector field \( f(x) \) is \( \nabla f = \frac{\partial f}{\partial x} \), where \( (\nabla f)_j = \frac{\partial f_i}{\partial x_j} \)

- Lie derivative of \( h \) with respect to \( f \) is a scalar function defined by
  \[
  L_f h = \nabla h f, \quad \text{where} \quad h : \mathbb{R}^n \rightarrow \mathbb{R} \text{ is a smooth scalar and} \ f : \mathbb{R}^n \rightarrow \mathbb{R}^n \text{ is a smooth vector field.}
  \]

- If \( g \) is another vector field: \( L_g L_f h = \nabla (L_f h) g \)
**Example:** For single output system \( \dot{x} = f(x), \ y = h(x) \) then

\[
\dot{y} = \frac{\partial h}{\partial x} \dot{x} = L_f h
\]

\[
\ddot{y} = \frac{\partial [L_f h]}{\partial x} \dot{x} = L_f^2 h
\]

- If \( V \) is a Lyap. fcn candidate, its derivative \( \dot{V} \) can be written as \( L_f V \).

- **Lie bracket of \( f \) and \( g \)** is a third vector field defined by \([f, g] = \nabla g f - \nabla f g\), where \( f \) and \( g \) two vector field on \( R^n \).

- The Lie bracket \([f, g]\) is also written as \( ad_f g \) (where ad stands for "adjoint").

**Example:** Consider \( \dot{x} = f(x) + g(x)u \) where

\[
f = \begin{bmatrix}
-2x_1 + ax_2 + \sin x_1 \\
-x_2 \cos x_1
\end{bmatrix}, \quad g = \begin{bmatrix}
0 \\
\cos(2x_1)
\end{bmatrix}
\]

- So the Lie bracket is:

\[
[f, g] = \begin{bmatrix}
-a \cos(2x_1) \\
\cos x_1 \cos(2x_1) - 2 \sin(2x_1)(-2x_1 + ax_2 + \sin x_1)
\end{bmatrix}
\]
Lemma: Lie brackets have the following properties:

1. Bilinearity:

\[ [\alpha_1 f_1 + \alpha_2 f_2, g] = \alpha_1 [f_1, g] + \alpha_2 [f_2, g] \]

\[ [f, \alpha_1 g_1 + \alpha_2 g_2] = \alpha_1 [f, g_1] + \alpha_2 [f, g_2] \]

where \( f, f_1, f_2, g, g_1, g_2 \) are smooth vector fields and \( \alpha_1 \) and \( \alpha_2 \) are constant scalars.

2. Skew-commutativity:

\[ [f, g] = -[g, f] \]

3. Jacobi identity

\[ L_{ad_f} g h = L_f L_g h - L_g L_f h \]

where \( h \) is a smooth fcn.
Diffeomorphism

- The concept of diffeomorphism can be applied to transform a nonlinear system into another nonlinear system in terms of a new set of states.

- **Definition:** A function $\phi : \mathbb{R}^n \to \mathbb{R}^n$ defined in a region $\Omega$ is called a diffeomorphism if it is smooth, and if its inverse $\phi^{-1}$ exists and is smooth.

- If the region $\Omega$ is the whole space $\mathbb{R}^n$, $\phi(x)$ is a global diffeomorphism.

- Global diffeomorphisms are rare, we are looking for local diffeomorphisms.

- **Lemma:** Let $\phi(x)$ be a smooth function defined in a region $\Omega$ in $\mathbb{R}^n$. If the Jacobian matrix $\nabla \phi$ is non-singular at a point $x = x_0$ of $\Omega$, then $\phi(x)$ defines a local diffeomorphism in a subregion of $\Omega$. 
Diffeomorphism

Consider the dynamic system described by

\[ \dot{x} = f(x) + g(x)u, \quad y = h(x) \]

Let the new set of states \( z = \phi(x) \rightarrow \dot{z} = \frac{\partial \phi}{\partial x} \dot{x} = \frac{\partial \phi}{\partial x} (f(x) + g(x)u) \)

The new state-space representation

\[ \dot{z} = f^*(z) + g^*(z)u, \quad y = h^*(z) \]

where \( x = \phi^{-1}(z) \).

**Example of a non-global diffeomorphism:** Consider

\[
\begin{bmatrix}
  z_1 \\
  z_2
\end{bmatrix} = \phi(x) = \begin{bmatrix}
  2x_1 + 5x_1x_2^2 \\
  3 \sin x_2
\end{bmatrix}
\]

Its Jacobian matrix:

\[
\begin{bmatrix}
  2 + 5x_2^2 & 10x_1x_2 \\
  0 & 3 \cos x_2
\end{bmatrix}.
\]

rank is 2 at \( x = (0, 0) \rightarrow \) local diffeomorphism around the origin where \( \Omega = \{(x_1, x_2), |x_2| < \pi/2\} \).

outside the region, the inverse of \( \phi \) does not uniquely exist.
Frobenius Theorem

- An important tool in feedback linearization
- Provide necessary and sufficient conditions for solvability of PDEs.
- Consider a PDE with \(n=3\):

\[
\frac{\partial h}{\partial x_1} f_1 + \frac{\partial h}{\partial x_2} f_2 + \frac{\partial h}{\partial x_3} f_3 = 0 \\
\frac{\partial h}{\partial x_1} g_1 + \frac{\partial h}{\partial x_2} g_2 + \frac{\partial h}{\partial x_3} g_3 = 0
\] (7)

where \(f_i(x_1, x_2, x_3), \ g_i(x_1, x_2, x_3) \ (i = 1, 2, 3)\) are known scalar functions and \(h(x_1, x_2, x_3)\) is an unknown function.

- This set of PDEs is uniquely determined by the two vectors \(f = [f_1 \ f_2 \ f_3]^T\), \(g = [g_1 \ g_2 \ g_3]^T\).

- If the solution \(h(x_1, x_2, x_3)\) exists, the set of vector fields \(\{f, g\}\) is completely integrable.

- When the equations are solvable?
Frobenius Theorem

- Frobenius theorem states that Equation (7) has a solution \( h(x_1, x_2, x_3) \) iff there exists **scalar functions** \( \alpha_1(x_1, x_2, x_3) \) and \( \alpha_2(x_1, x_2, x_3) \) such that

\[
[f, g] = \alpha_1 f + \alpha_2 g
\]

i.e., if the Lie bracket of \( f \) and \( g \) can be expressed as a linear combination of \( f \) and \( g \)

- This condition is called **involutivity of the vector fields** \( \{f, g\} \).

- Geometrically, it means that the vector field \([f, g]\) is in the plane formed by the two vectors \( f \) and \( g \)

- The set of vector fields \( \{f, g\} \) is completely integrable iff it is involutive.

- **Definition (Complete Integrability):** A **linearly independent set of vector fields** \( \{f_1, f_2, ..., f_m\} \) on \( \mathbb{R}^n \) is said to be completely integrable, iff, there exist \( n - m \) scalar fcns \( h_1(x), h_2(x), ..., h_{n-m}(x) \) satisfying the system of PDEs:

\[
\nabla h_i \cdot f_j = 0
\]

where \( 1 \leq i \leq n-m, \ 1 \leq j \leq m \) and \( \nabla h_i \) are linearly independent.
Number of vectors: \( m \), dimension of the vectors: \( n \), number of unknown scalar fcns \( h_i: (n-m) \), number of PDEs: \( m(n-m) \)

**Definition (Involutivity):** A linearly independent set of vector fields \( \{ f_1, f_2, \ldots, f_m \} \) on \( \mathbb{R}^n \) is said to be involutive iff, there exist scalar fcns \( \alpha_{ijk}: \mathbb{R}^N \to \mathbb{R} \) s.t.

\[
[f_i, f_j](x) = \sum_{k=i}^{m} \alpha_{ijk}(x) f_k(x) \quad \forall \ i, j
\]

i.e., the Lie bracket of any two vector fields from the set \( \{ f_1, f_2, \ldots, f_m \} \) can be expressed as the linear combination of the vectors from the set.

- Constant vector fields are involutive since their Lie brackets are zero
- A set composed of a single vector is involutive:

\[
[f, f] = (\nabla f)f - (\nabla f)f = 0
\]

- Involutivity means:

\[
\text{rank} (f_1(x) \ldots f_m(x)) = \text{rank} (f_1(x) \ldots f_m(x) [f_i, f_j](x))
\]

for all \( x \) and for all \( i, j \).
Frobenius Theorem

Theorem (Frobenius): Let $f_1$, $f_2$, ..., $f_m$ be a set of linearly independent vector fields. The set is completely integrable iff it is involutive.

Example: Consider the set of PDEs:

$$4x_3 \frac{\partial h}{\partial x_1} - \frac{\partial h}{\partial x_2} = 0$$
$$-3x_1 \frac{\partial h}{\partial x_1} + (-4x_3^2 - 3x_2) \frac{\partial h}{\partial x_2} + 2x_3 \frac{\partial h}{\partial x_3} = 0$$

The associated vector fields are $\{f_1, f_2\}$

$$f_1 = [4x_3 - 1 \ 0]^T \quad f_2 = [-3x_1 (-4x_3^2 - 3x_2) \ 2x_3]^T$$

We have $[f_1, f_2] = [-12x_3 \ 3 \ 0]^T$

Since $[f_1, f_2] = -3f_1 + 0f_2$, the set $\{f_1, f_2\}$ is involutive and the set of PDEs are solvable.
Input-State Linearization

Consider the following SISO nonlinear system

\[ \dot{x} = f(x) + g(x)u \]  

(8)

where \( f \) and \( g \) are smooth vector fields

The above system is also called “linear in control” or “affine”

If we deal with the following class of systems:

\[ \dot{x} = f(x) + g(x)w(u + \phi(x)) \]

where \( w \) is an invertible scalar fcn and \( \phi \) is an arbitrary fcn

- We can use \( v = w(u + \phi(x)) \) to get the form (8).
- Control design is based on \( v \) and \( u \) can be obtained by inverting \( w \):
  \[ u = w^{-1}(v) - \phi(x) \]

Now we are looking for

- Conditions for system linearizability by an input-state transformation
- A technique to find such transformations
- A method to design a controller based on such linearization technique
Input-State Linearization

**Definition: Input-State Linearization** The nonlinear system (8) where \( f(x) \) and \( g(x) \) are smooth vector fields in \( \mathbb{R}^n \) is input-state linearizable if there exist region \( \Omega \) in \( \mathbb{R}^n \), a diffeomorphism mapping \( \phi : \Omega \rightarrow \mathbb{R}^n \), and a control law:

\[
    u = \alpha(x) + \beta(x)v
\]

s.t. new state variable \( z = \phi(x) \) and new input variable \( v \) satisfy an LTI relation:

\[
    \dot{z} = Az + Bv
\]

\[
    A = \begin{bmatrix}
    0 & 1 & 0 & \ldots & 0 \\
    0 & 0 & 1 & \ldots & . \\
    . & . & . & \ldots & . \\
    . & . & . & \ldots & . \\
    0 & 0 & 0 & \ldots & 1 \\
    0 & 0 & 0 & \ldots & 0
\end{bmatrix}
\]

\[
    B = \begin{bmatrix}
    0 \\
    0 \\
    \vdots \\
    0 \\
    1
\end{bmatrix}
\]  

(9)

The new state \( z \) is called the **linearizing state** and the control law \( u \) is called the **linearizing control law**

Let \( z = z(x) \)
(9) is the so-called controllability or companion form

The companion form can be obtained from any form by a transformation \( \Rightarrow \) the above form is a general form.

This form is a special case of Input-Output linearization leading to relative degree \( r = n \).

Hence, if the system I-O linearizable with \( r = n \), it is also I-S linearizable.

On the other hand, if the system is I-S linearizable, it is also I-O linearizable with \( y = z, \ r = n \).

**Lemma:** An \( n^{th} \) order nonlinear system is I-S linearizable iff there exists a scalar fcn \( z_1(x) \) for which the system is I-O linearizable with \( r = n \).

**Conditions for Input-State Linearization:**

**Theorem:** The nonlinear system (8) with \( f(x) \) and \( g(x) \) being smooth vector field is input-state linearizable iff there exists a region \( \Omega \) s.t. the following conditions hold:

- The vector fields \( \{g, \ ad_f g, \ ... \ ad_f^{n-1} g\} \) are linearly independent in \( \Omega \)
- The set \( \{g, \ ad_f g, \ ... \ ad_f^{n-2} g\} \) is involutive in \( \Omega \)
The first condition can be interpreted as a controllability condition.

For linear system, the vector field above becomes \(\{B, AB, \ldots A^{n-1}B\}\).

Linear independency is equivalent to invertibility of controllability matrix.

The second condition is always satisfied for linear systems since the vector fields are constant, but for nonlinear system is not necessarily satisfied.

It is necessary according to Frobenius theorem for existence of \(z_1(x)\).

**Lemma:** If \(z(x)\) is a smooth vector field in \(\Omega\), then the set of equations

\[
L_g z = L_g L_f z = \ldots L_g L_f^k z = 0
\]

is equivalent to

\[
L_g z = L_{ad} f g z = \ldots L_{ad} f^k g z = 0
\]

**Proof:**

Let \(k = 1\), from Jacobi’s identity, we have

\[
L_{ad} f g z = L_f L_g z - L_g L_f z = 0 - 0 = 0
\]
When $k = 2$, we have from Jacobi’s identity:

$$L_{ad_f^2 g z} = L_f^2 L_g z - 2L_f L_g L_f z + L_g L_f^2 z = 0 - 0 + 0 = 0$$

Proof of the linearization theorem:

Necessity:

Suppose state transformation $z = z(x)$ and input transformation $u = \alpha(x) + \beta(x)v$ s.t. $z$ and $v$ satisfy (9), i.e.

$$\dot{z}_1 = \frac{\partial z_1}{\partial x} (f + gu) = z_2$$

similarly:

$$\frac{\partial z_1}{\partial x} f + \frac{\partial z_1}{\partial x} gu = z_2$$
$$\frac{\partial z_2}{\partial x} f + \frac{\partial z_2}{\partial x} gu = z_3$$
$$\vdots$$
$$\frac{\partial z_n}{\partial x} f + \frac{\partial z_n}{\partial x} gu = v$$
\( z_1, \ldots, z_n \) are independent of \( u \), but \( v \) is not, hence:

\[

g z_1 = g z_2 = \ldots g z_{n-1} = 0, \quad g z_n \neq 0 \\
\mathbf{f} z_i = z_{i+1}, \quad i = 1, 2, \ldots, n-1
\]

Use, \( \mathbf{z} = [z_1 \quad \mathbf{f} z_1, \ldots \quad \mathbf{f}^{n-1} z_1]^T \) to get

\[
\dot{z}_k = z_{k+1}, \quad k = 1, \ldots, n-1 \\
\dot{z}_n = \mathbf{f}^{n} z_1 + g \mathbf{f}^{n-1} z_1 u
\]

The above equations can be expressed in terms of \( z_1 \) only

\[
\nabla z_1 \text{ad}_f^k g = 0, \quad k = 0, 1, 2, \ldots, n-2 \\
\nabla z_1 \text{ad}_f^{n-1} g = (-1)^{n-1} g \mathbf{f}
\]

First note that for above eqs to hold, the vector field \( g, \text{ad}_f g, \ldots, \text{ad}_f^{n-1} g \) must be linearly independent.

If for some \( i (i \leq n-1) \) there exist scalar fcns \( \alpha_1(x), \ldots, \alpha_{i-1}(x) \) s.t.

\[
\text{ad}_f^i g = \sum_{k=0}^{i-1} \alpha_k \text{ad}_f^k g
\]
We, then have:

\[ \therefore \text{ad}_f \, ^{n-1}g = \sum_{k=n-i-1}^{n-2} \alpha_k \text{ad}_f \, ^kg \]

\[ \therefore \nabla z_1 \text{ad}_f \, ^{n-1}g = \sum_{k=n-i-1}^{n-2} \alpha_k \nabla z_1 \text{ad}_f \, ^kg = 0 \quad (12) \]

\[ \therefore \text{Contradicts with (11).} \]

The second property is that \( \exists \) a scalar fcn \( z_1 \) that satisfy \( n-1 \) PDEs

\[ \nabla z_1 \text{ad}_f \, ^kg = 0 \]

\[ \therefore \text{From the necessity part of Frobenius theorem, we conclude that the set of vector field must be involutive.} \]

**Sufficient condition**

Involutivity condition \( \iff \) Frobenius theorem, \( \exists \) a scalar fcn \( z_1(x) \):

\[ L_g z_1 = L_{ad_f} g z_1 = \ldots L_{ad_f} \, ^kg z_1 = 0, \quad \text{implying} \]

\[ L_g z_1 = L_g L_f z_1 = \ldots L_g L_f \, ^kz_1 = 0 \]
Define the new sets of variable as 
\[ z = [z_1 \ L_f z_1 \ ... \ L_f \ n^{-1} z_1]^T, \]
to get
\[ \dot{z}_k = z_{k+1} \quad k = 1, \ldots, n - 1 \]
\[ \dot{z}_n = L_f \ n z_1 + L_g L_f \ n^{-1} z_1 u \quad (13) \]

The question is whether \( L_g L_f \ n^{-1} z_1 \) can be equal to zero.

Since \( \{g, \text{ad}_f g, \ldots, \text{ad}_f \ n^{-1} g\} \) are linearly independent in \( \Omega \):
\[ L_g L_f \ n^{-1} z_1 = (-1)^{n-1} L_{\text{ad}_f} \ n^{-1} g z_1 \]

We must have \( L_{\text{ad}_f} \ n^{-1} g z_1 \neq 0 \), otherwise the nonzero vector \( \nabla z_1 \) satisfies
\[ \nabla z_1 \ [g, \text{ad}_f g, \ldots, \text{ad}_f \ n^{-1} g] = 0 \]
i.e. \( \nabla z_1 \) is normal to \( n \) linearly independent vector \( \implies \) impossible

Now, we have:
\[ \dot{z}_n = L_f \ n z_1 + L_g L_f \ n^{-1} z_1 u = a(x) + b(x) u \]
Now, select \( u = \frac{1}{b(x)}(-a(x) + v) \) to get:
\[
\dot{z}_n = v
\]
implying input-state linearization is obtained. □

Summary: how to perform input-state Linearization

1. Construct the vector fields \( g, \, ad_f g, \, ... \, ad_f^{n-1}g \)
2. Check the controllability and involutivity conditions
3. If the conditions hold, obtain the first state \( z_1 \) from:
   \[
   \nabla z_1 ad_f^i g = 0 \quad i = 0, \ldots, n-2
   \]
   \[
   \nabla z_1 ad_f^{n-1}g \neq 0
   \]
4. Compute the state transformation \( z(x) = [z_1 \, L_f z_1 \, \ldots \, L_f^{n-1}z_1]^T \) and the input transformation \( u = \alpha(x) + \beta(x)v \):
   \[
   \alpha(x) = -\frac{L_f^n z_1}{L_g L_f^{n-1} z_1}
   \]
   \[
   \beta(x) = \frac{1}{L_g L_f^{n-1} z_1}
   \]
Example: A single-link flexible-joint manipulator:

- The link is connected to the motor shaft via a torsional spring
- **Equations of motion:**
  
  \[
  \begin{align*}
  l \ddot{q}_1 + MgL \sin q_1 + K(q_1 - q_2) &= 0 \\
  J \ddot{q}_2 - K(q_1 - q_2) &= u
  \end{align*}
  \]

- nonlinearities appear in the first equation and torque is in the second equation
- Let:

  \[
  x = \begin{bmatrix}
  q_1 \\
  \dot{q}_1 \\
  q_2 \\
  \dot{q}_2
  \end{bmatrix}, \quad
  f = \begin{bmatrix}
  x_2 \\
  -\frac{MgL}{I} \sin x_1 - \frac{K}{I}(x_1 - x_3) \\
  x_4 \\
  \frac{K}{J}(x_1 - x_3)
  \end{bmatrix}, \quad
  g = \begin{bmatrix}
  0 \\
  0 \\
  0 \\
  \frac{1}{J}
  \end{bmatrix}
  \]

- **Controllability and involutivity conditions:**

  \[
  \begin{bmatrix}
  g \\
  adf g \\
  adf^2 g \\
  adf^3 g
  \end{bmatrix}
  = \begin{bmatrix}
  0 & 0 & 0 & -\frac{K}{IJ} \\
  0 & 0 & \frac{K}{IJ} & 0 \\
  0 & -\frac{1}{J} & 0 & -\frac{K}{J^2} \\
  -\frac{1}{J} & 0 & -\frac{K}{J^2} & 0
  \end{bmatrix}
  \]
Example: Cont’d

- It’s full rank for \( k > 0 \) and \( IJ < \infty \) \( \implies \) vector fields are linearly independent
- Vector fields are constant \( \implies \) involutive
- The system is input-state linearizable
- Computing \( z = z(x) \), \( u = \alpha(x) + \beta(x)v \)
- \( \frac{\partial z_1}{\partial x_2} = 0, \frac{\partial z_1}{\partial x_3} = 0, \frac{\partial z_1}{\partial x_4} = 0, \frac{\partial z_1}{\partial x_1} \neq 0 \)
- Hence, \( z_1 \) is the fcn of \( x_1 \) only. Let \( z_1 = x_1 \), then

\[
\begin{align*}
    z_2 &= \nabla z_1 f = x_2 \\
    z_3 &= \nabla z_2 f = -\frac{MgL}{l} \sin x_1 - \frac{K}{l} (x_1 - x_3) \\
    z_4 &= \nabla z_3 f = -\frac{MgL}{l} x_2 \cos x_1 - \frac{K}{l} (x_2 - x_4)
\end{align*}
\]
Example: Cont’d

The input transformation is given by:

\[ u = \frac{(v - \nabla z_4 f)}{(\nabla z_4 g)} = \frac{IJ}{K} (v - a(x)) \]

\[ a(x) = \frac{MgL}{I} \sin x_1 (x_2^2 + \frac{MgL}{I} \cos x_1 + \frac{K}{l}) \]
\[ + \frac{K}{l} (x_1 - x_3) \left( \frac{K}{J} + \frac{K}{l} + \frac{MgL}{I} \cos x_1 \right) \]

As a result, we get the following set of linear equations

\[ \dot{z}_1 = z_2, \quad \dot{z}_2 = z_3 \]
\[ \dot{z}_3 = z_4, \quad \dot{z}_4 = v \]

The inverse of the state transformation is given by:

\[ x_1 = z_1, \quad x_2 = z_2 \]
\[ x_3 = z_1 + \frac{I}{K} \left( z_3 + \frac{MgL}{l} \sin z_1 \right) \]
\[ x_4 = z_2 + \frac{I}{K} \left( z_4 + \frac{MgL}{l} z_2 \cos z_1 \right) \]
Input-State Linearization

- State and input transformations are defined globally.
- In this example, transformed state have physical meaning, $z_1$ : link position, $z_2$ : link velocity, $z_3$ : link acceleration, $z_4$ : link jerk.
- It could be obtained by I-O linearization, i.e. by differentiating the output $q_1$.
- We can transform the inequality (11) to a normalized equation by setting $\nabla z_1 a_d f^n - 1 g = 1$ resulting in:

\[
\begin{bmatrix}
\frac{\partial z_1}{\partial x_1} \\
\vdots \\
\frac{\partial z_1}{\partial x_n}
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
1
\end{bmatrix}
\]
Control Design

- Once, the linearized dynamics is obtained, either a tracking or stabilization problem can be solved.
- For instance, in flexible-joint manipulator case, we have
  \[ z_1^{(4)} = v \]

- Then, a tracking controller can be obtained as
  \[ v = z_{d1}^{(4)} - a_3 \ddot{z}_1^{(3)} - a_2 \dddot{z}_1 - a_1 \dot{z}_1 - a_0 \ddot{z}_1 \]
  where \( \tilde{z}_1 = z_1 - z_{d1} \).

- The error dynamics is then given by:
  \[ \dddot{z}_1^{(4)} + a_3 \dddot{z}_1^{(3)} + a_2 \dddot{z}_1 + a_1 \dot{z}_1 + a_0 \ddot{z}_1 = 0 \]

- The above dynamics is exponentially stable if \( a_i \) are selected s.t.
  \[ s_4 + a_3 s^3 + a_2 s^2 + a_1 s + a_0 \] is Hurwitz.
Consider the system:

\[ \dot{x} = f(x) + g(x)u \]
\[ y = h(x) \]  

(14)

Input-output linearization yields a linear relationship between the input output \( y \) and the input \( v \) (similar to \( v \) in I-S Lin.)

- How to generate a linear I-O relation for such systems?
- What are the internal dynamics and zero-dynamics associated with this I-O linearization?
- How to design a stable controller based on this technique?

**Performing I-O Linearization**

- The basic approach is to differentiate the output \( y \) until the input \( u \) appears, then design \( u \) to cancel nonlinearities.
- Sometime, cancelation might not be possible due to the undefined relative degree.
Well Defined Relative Degree

- Differentiate $y$ and express it in the form of Lie derivative:
  \[
  \dot{y} = \nabla h(f + gu) = L_f h(x) + L_g h(x)u
  \]

  if $L_g h(x) \neq 0$ for some $x = x_0$ in $\Omega_x$, then continuity implies that $L_g h(x) \neq 0$ in some neighborhood $\Omega$ of $x_0$. Then, the input transformation
  \[
  u = \frac{1}{L_g h(x)}(-L_f h(x) + v)
  \]
  results in a linear relationship between $y$ and $v$, namely $\dot{y} = v$.

- If $L_g h(x) = 0$ for all $x \in \Omega_x$, differentiate $\dot{y}$ to obtain
  \[
  \ddot{y} = L_f^2 h(x) + L_g L_f h(x)u
  \]

- If $L_g L_f h(x) = 0$ for all $x \in \Omega_x$, keep differentiating until for some integer $r$, $L_g L_f \, r^{-1} h(x) \neq 0$ for some $x = x_0 \in \Omega_x$.
Hence, we have

\[ y^{(r)} = L_f^r h(x) + L_g L_f^{r-1} h(x) u \]  

(15)

and the control law

\[ u = \frac{1}{L_g L_f^{r-1} h(x)} (-L_f^r h(x) + v) \]

yields a linear mapping:

\[ y^{(r)} = v \]

The number \( r \) of differentiation required for \( u \) to appear is called the relative degree of the system.

\( r \leq n \), if \( r = n \), the input-state realization is obtained with \( z_1 = y \).

**Definition:** *The SISO system is said to have a relative degree \( r \) in \( \Omega \) if:

\[ L_g L_f^i h(x) = 0 \quad 0 \leq i \leq r - 2 \]

\[ L_g L_f^{r-1} h(x) \neq 0 \]*
Undefined Relative Degree

- Sometimes, we are interested in the properties of a system about a specific operating point $x_0$.
- Then, we say the system has relative degree $r$ at $x_0$ if
  \[ L_g L_f \left( r^{-1} h(x_0) \right) \neq 0 \]
- However, it might happen that $L_g L_f \left( r^{-1} h(x) \right)$ is zero at $x_0$, but nonzero in a close neighborhood of $x_0$.
- The relative degree of the nonlinear system is then undefined at $x_0$.
- **Example:**

\[ \ddot{x} = \rho(x, \dot{x}) + u \]

where $\rho$ is a smooth nonlinear fcn. Define $x = [x \ \dot{x}]^T$ and let $y = x \implies$ the system is in companion form with $r = 2$. 
However, if we define $y = x^2$, then:

\[
\dot{y} = 2x\dot{x} \\
\ddot{y} = 2x\ddot{x} + 2\dot{x}^2 = 2x\rho(x, \dot{x}) + 2xu + 2\dot{x}^2 \implies L_g L_f h = 2x
\] (16)

The system has neither relative degree 1 nor 2 at $x_0$.

Sometime, change of output leads us to a solvable problem.

We assume that the relative degree is well defined.

**Normal Forms**

- When, the relative degree is defined as $r \leq n$, using $y, \dot{y}, ..., y^{(r-1)}$, we can transform the system into the so-called normal form.
- Norm form allows a formal treatment of the notion of internal dynamics and zero dynamics.
- Let

\[
\mu = [\mu_1 \mu_2 ... \mu_r]^T = [y \dot{y} ... y^{(r-1)}]^T
\]

in a neighborhood $\Omega$ of a point $x_0$. 
Normal Form

The normal form of the system can be written as

\[
\begin{bmatrix}
\mu_2 \\
\vdots \\
\mu_r \\
a(\mu, \Psi) + b(\mu, \Psi)u
\end{bmatrix} = \dot{\mu}
\]

(17)

\[
\dot{\Psi} = w(\mu, \Psi) 
\]

(18)

The \( \mu_i \) and \( \Psi_j \) are called *normal coordinate* or *normal states*.

The first part of the Normal form, (17) is another form of (15), however in (18) the input \( u \) does not appear.

The system can be transformed to this form if the state transformation \( \phi(x) \) is a local diffeomorphism: \( \phi(\mu_1 \ldots \mu_r \ \Psi_1 \ldots \Psi_{n-r})^T \)

To show that \( \phi \) is a diffeomorphism, we must show that the Jacobian is invertible, i.e. \( \nabla \mu_j \) and \( \nabla \Psi_i \) are all linearly independent.
Normal Form

- $\nabla \mu_j$ are linearly independent $\iff \mu$ can be part of state variables, ($\mu$ is output and its $r - 1$ derivatives)
- There exist $n - r$ other vector fields that complete the transformation
- Note that $u$ does not appear in (17), hence:
  \[ \nabla \psi_j g = 0 \quad 1 \leq j \leq n - r \]
  \[ \therefore \psi \text{ can be obtained by solving } n - r \text{ PDE above.} \]
- Generally, internal dynamics can be obtained simpler by intuition.
- **Zero Dynamics**
  - System dynamics into two parts:
    1. external dynamics $\dot{\mu}$
    2. internal dynamics $\Psi$
  - For tracking problems ($y \longrightarrow y_d$), one can easily design $v$ once the linear relation is obtained.
  - The question is whether the internal dynamics remain bounded
Zero-Dynamics

- Stability of the zero dynamics (i.e. internal dynamics when $y$ is kept 0) gives an idea about the stability of internal dynamics
- $u$ is selected s.t. $y$ remains zero at all time.

\[
y^{(r)}(t) = L_f^r h(x) + L_g L_f^{r-1} h(x) u_0 \equiv 0 \implies u_0(x) = -\frac{L_f^r h(x)}{L_g L_f^{r-1} h(x)}
\]

- In normal form:

\[
\begin{cases}
\dot{\mu} = 0 \\
\dot{\Psi} = w(0, \Psi)
\end{cases}
\]

\[
u_0(\Psi) = \frac{-a(0, \Psi)}{b(0, \Psi)}
\]
Example

Consider

\[
\begin{align*}
\dot{x} &= \begin{bmatrix} -x_1 \\ 2x_1x_2 + \sin x_2 \\ 2x_2 \end{bmatrix} + \begin{bmatrix} e^{2x_2} \\ 1/2 \\ 0 \end{bmatrix} u \\
y &= h(x) = x_3
\end{align*}
\]

We have

\[
\begin{align*}
\dot{y} &= 2x_2 \\
\ddot{y} &= 2\dot{x}_2 = 2(2x_1x_2 + \sin x_2) + u
\end{align*}
\]

The system has relative degree \( r = 2 \) and

\[
\begin{align*}
L_f h(x) &= 2x_2 \\
L_g h(x) &= 0 \\
L_g L_f h(x) &= 1
\end{align*}
\]
Example Cont’d

- To obtain the normal form

\[
\mu_1 = h(x) = x_3 \\
\mu_2 = L_f h(x) = 2x_2
\]

- The third function \( \Psi(x) \) is obtained by

\[
L_g \Psi = \frac{\partial \psi}{\partial x_1} e^{2x_2} + \frac{1}{2} \frac{\partial \psi}{\partial x_2} = 0
\]

- One solution is \( \Psi(x) = 1 + x_1 - e^{2x_2} \)

- Consider the jacobian of state transformation \( z = [\mu_1 \mu_2 \Psi]^T \). The Jacobian matrix is

\[
\begin{bmatrix}
0 & 0 & 1 \\
0 & 2 & 0 \\
1 & -2e^{2x_2} & 0
\end{bmatrix}
\]
Example Cont’d

► The Jacobian is non-singular for any $x$. In fact, inverse transformation is given by:

\[
\begin{align*}
    x_1 & = -1 + \Psi + e^{\mu_2} \\
    x_2 & = \frac{1}{2} \mu_2 \\
    x_3 & = \mu_1
\end{align*}
\]

► State transformation is valid globally and the normal form is given by:

\[
\begin{align*}
    \dot{\mu}_1 & = \mu_2 \\
    \dot{\mu}_2 & = 2(-1 + \Psi + e^{\mu_2})\mu_2 + 2\sin(\mu_2/2) + u \\
    \dot{\Psi} & = (1 - \Psi - e^{\mu_2})(1 + 2\mu_2 e^{\mu_2}) - 2\sin(\mu_2/2)e^{\mu_2}
\end{align*}
\]

(20)

► Zero dynamics is obtained by setting $\mu_1 = \mu_2 = 0 \implies$

\[
\dot{\Psi} = -\Psi
\]

(21)
Zero-Dynamics

- In order to obtain the zero dynamics, it is not necessary to put the system into normal form.
- Since $\mu$ is known, we can intuitively find $n - r$ vector to complete the transformation.
- As mentioned before, zero dynamics is obtained by substituting $u_0$ for $u$ in internal dynamics.
- **Definition:** A nonlinear system with asymptotically stable zero dynamics is called asymptotically **minimum phase**.
- If the zero dynamics is stable for all $x$, the system is globally minimum phase, otherwise the results are local.
Consider again the nonlinear system

\[
\dot{x} = f(x) + g(x)u \\
y = h(x)
\]

Assume that the system is I-O linearized, i.e.

\[
y^{(r)} = L_f^r h(x) + L_g L_f^{r-1} h(x)u
\]

and the control law

\[
u = \frac{1}{L_g L_f^{r-1} h(x)} (-L_f^r h(x) + v)
\]

yields a linear mapping:

\[
y^{(r)} = v
\]

Now let \( v \) be chosen as

\[
v = -k_{r-1} y^{(r-1)} - ... - k_1 \dot{y} - k_0 y
\]

where \( k_i \) are selected s.t. \( K(s) = s^r + k_{r-1}s^{r-1} + ... + k_1 s + k_0 \) is Hurwitz
Then, provided that the zero-dynamics is asymptotically stable, the control law (24) and (25) locally stabilize the whole system:

**Theorem:** Suppose the nonlinear system (22) has a well defined relative degree \( r \) and its associated zero-dynamics is locally asymptotically stable. Now, if \( k_i \) are selected s.t. \( K(s) = s^r + k_{r-1}s^{r-1} + \ldots + k_1s + k_0 \) is Hurwitz, then the control law (24) and (25) yields a locally asymptotically stable system.

**Proof:** First, write the closed-loop system in a normal form:

\[
\dot{\mu} = \begin{bmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1 \\
-k_0 & -k_1 & -k_2 & \ldots & -k_{r-1}
\end{bmatrix} = A\mu
\]

\[
\dot{\psi} = w(\mu, \psi) = A_1\mu + A_2\psi + h.o.t.
\]

The above Eq. can be written as:

\[
\frac{d}{dt} \begin{bmatrix} \mu \\ \psi \end{bmatrix} = \begin{bmatrix} A & 0 \\ A_1 & A_2 \end{bmatrix} \begin{bmatrix} \mu \\ \psi \end{bmatrix} + h.o.t.
\]
Now, since the zero dynamics is asymptotically stable, its linearization $\dot{\Psi} = A_2 \Psi$ is either asymptotically stable or marginally stable.

- If $A_2$ is **asymptotically stable**, then all eigenvalues of the above system matrix are in LHP and the linearized system is stable and the nonlinear system is locally asymptotically stable.

- If $A_2$ is **marginally stable**, asymptotic stability of the closed-loop system was shown in (Byrnes and Isidori, 1988).

For stabilization where state convergence is required, we can freely choose $y = h(x)$ to affect zero-dynamics.

**Example**: Consider the nonlinear system:

\[
\begin{align*}
\dot{x}_1 &= x_1^2 x_2 \\
\dot{x}_2 &= 3x_2 + u
\end{align*}
\]

System linearization at $x = 0$:

\[
\begin{align*}
\dot{x}_1 &= 0 \\
\dot{x}_2 &= 3x_2 + u
\end{align*}
\]

thus has an uncontrollable mode.
Example (cont’d)

Define \( y = -2x_1 - x_2 \)

\[
\dot{y} = -2\dot{x}_1 - \dot{x}_2 = -2x_1^2x_2 - 3x_2 - u
\]

Hence, the relative degree \( r = 1 \) and the associated zero-dynamics is

\[
\dot{x}_1 = -2x_1^3
\]

The zero-dynamics is asymptotically stable, hence the control law \( u = -2x_1^2x_2 - 4x_2 - 2x_1 \) locally stabilizes the system

Global Asymptotic Stabilization

Stability of the zero-dynamics only guarantees local stability unless relative degree is \( n \) in which case there is no internal dynamics

Can the idea of I-O linearization be used for global stabilization problem?

Can the idea of I-O linearization be used for systems with unstable zero dynamics?
Global Asymptotic Stabilization

- Global stabilization approach based on partial feedback linearization is to simply regard the problem as a standard Lyapunov controller design problem.

- **But** simplified by the fact that in normal form part of the system dynamics is now linear.

- The basic idea is to view $\mu$ as the input to the internal dynamics and $\Psi$ as its output.
  - The first step: find the control law $\mu_0 = \mu_0(\Psi)$ which stabilizes the internal dynamics with the corresponding Lyapunov fcn $V_0$.
  - Then: find a Lyapunov fcn candidate for the whole system (as a modified version of $V_0$) and choose the control input $v$ s.t. $V$ be a Lyapunov fcn for the whole closed-loop dynamics.
Example:

- Consider a nonlinear system with the normal form:

\[
\begin{align*}
\dot{y} &= v \\
\ddot{z} + \dot{z}^3 + yz &= 0 \\
\end{align*}
\]  

(26)

where \( v \) is the control input and \( \Psi = [z \ \dot{z}]^T \)

- Considering \( y \) as an input to internal dynamics (26), it would be asymptotically stabilized by the choice of \( y = y_0 = z^2 \)
  - Let \( V_0 \) be a Lyap. fcn:

\[
V_0 = \frac{1}{2} \dot{z}^2 + \frac{1}{4} z^4
\]

- Differentiating \( V_0 \) along the actual dynamics results in

\[
\dot{V}_0 = -\dot{z}^4 - z\dot{z}(y - z^2)
\]
Consider the Lyap. fcn candidate, obtained by adding a quadratic “error” term in $y - y_0$ to $V_0$

$$V = V_0 + \frac{1}{2}(y - z^2)^2$$

$$\therefore \dot{V} = -\dot{z}^4 - (y - z^2)(v - 3z\dot{z})$$

The following choice of control action will then make $\dot{V}$ n.s.d.

$$v = -y + z^2 + 3z\dot{z}$$

$$\therefore \dot{V} = -\dot{z}^4 - (y - z^2)^2$$

Application of Invariant-set theorem shows all states converges to zero
Example: A non-minimum phase system

Consider the system dynamics

\[
\begin{align*}
y(t) &= v(t) \\
\dot{z}(t) + \dot{z}^3 - z^5 + yz &= 0
\end{align*}
\]

where again \(\Psi = [z \; \dot{z}]^T\)

- The system is non-minimum phase since its zero-dynamics is unstable
- The zero-dynamics would be stable if we select \(y = 2z^4\):

\[
V_0 = \frac{1}{2} \dot{z}^2 + \frac{1}{6} z^6 \implies \dot{V}_0 = -\dot{z}^4 - z\dot{z}(y - 2z^4)
\]

- Consider the Lyapunov function candidate

\[
V = V_0 + \frac{1}{2}(y - 2z^4)^2 \implies \dot{V} = -\dot{z}^4 - (y - 2z^4)(\nu - 8z^3\dot{z} - z\ddot{z})
\]

- Suggesting the following choice of control law

\[
\nu = -y + 2z^4 + 8z^3\dot{z} + z\ddot{z} \implies \dot{V} = -\dot{z}^4 - (y - 2z^4)^2
\]

- Application of Invariant-set theorem shows all states converges to zero
Tracking Control

- I/O linearization can be used in tracking problem
- Let \( \mu_d = [y_d \ y_d' \ ... \ y_d^{(r-1)}]^T \) and the tracking error \( \tilde{\mu}(t) = \mu(t) - \mu_d(t) \)
- **Theorem:** Assume the system (22) has a well defined relative degree \( r \) and \( \mu_d \) is smooth and bounded and that the solution \( \Psi_d: \)
  \[
  \dot{\Psi}_d = w(\mu_d, \Psi_d), \quad \Psi_d(0) = 0
  \]
  exists and bounded and is uniformly asymptotically stable. Choose \( k_i \) s.t \( K(s) = s^r + k_{r-1}s^{r-1} + \ldots + k_1s + k_0 \) is Hurwitz, then by using
  \[
  u = \frac{1}{L_g L_f \mu_1} \left[ -L_f \mu_1 + y_d^{(r)} - k_{r-1}\tilde{\mu}_r - \ldots - k_0\tilde{\mu}_r \right] \quad (27)
  \]
  the whole system remains bounded and the tracking error \( \tilde{\mu} \) converge to zero exponentially.
- **Proof:** Refer to Isidori (1989).
- For perfect tracking \( \mu(0) \equiv \mu_d(0) \)
Tracking Control for Non-minimum Phase Systems:

- The tracking control (27) cannot be applied to non-minimum phase systems since they cannot be inverted.
- Hence we cannot have perfect or asymptotic tracking and should seek controllers that yield small tracking errors.
- One approach is the so-called Output redefinition:
  - The new output $y_1$ is defined so that the associated zero-dynamics is stable.
  - $y_1$ is defined so that it is close to the original output $y$ in the frequency range of interest.
  - Then, tracking $y_1$ also implies good tracking of the original output $y$.

**Example:** Consider a linear system

$$y = \frac{(1 - \frac{s}{b}) B_0(s)}{A(s)} u \quad b > 0$$

- Perfect/asymptotic tracking is impossible due to the presence of zero @ $s = b$. 
Example Cont’d

Let us redefine the output as

\[ y_1 = \frac{B_0(s)}{A(s)} u \]

with the desired output for \( y_1 \) be simply \( y_d \)

A controller can be found s.t. \( y_1 \) asymptotically tracks \( y_d \). What about the actual tracking error?

\[ y(s) = \left(1 - \frac{s}{b}\right) y_1 = \left(1 - \frac{s}{b}\right) y_d \]

Thus, the tracking error is proportional to the desired velocity \( \dot{y}_d \):

\[ y(t) - y_d(t) = -\frac{\dot{y}_d(t)}{b} \]

Tracking error is bounded as long as \( \dot{y}_d \) is bounded, it is small when the frequency content of \( y_d \) is well below \( b \)
Example Cont’d

- An alternative output, motivated by \((1 - \frac{s}{b}) \approx 1/(1 + \frac{s}{b})\) for small \(|s|/b\):

\[
y_2 = \frac{B_0(s)}{A(s)(1 + \frac{s}{b})}u
\]

\[
y(s) = \left(1 - \frac{s}{b}\right) \left(1 + \frac{s}{b}\right) y_d = \left(1 - \frac{s^2}{b^2}\right) y_d
\]

- Thus, the tracking error is proportional to the desired acceleration \(\ddot{y}_d\):

\[
y(t) - y_d(t) = -\frac{\ddot{y}_d(t)}{b^2}
\]

- Small tracking error if the frequency content of \(y_d\) is below \(b\)
Another approximate tracking (Hauser, 1989) can be obtained by
- When performing I/O linearization, using successive differentiation, simply **neglect** the terms containing the input
- Keep differentiating $n$ timed (system order)
- Approximately, there is no zero dynamics
- It is meaningful if the coefficients of $u$ at the intermediate steps are “small” or the system is “weakly non-minimum phase” system
- The approach is similar to neglecting fast RHP zeros in linear systems.

Zero-dynamics is the property of the plant, choice of input and output and cannot be changed by feedback:
- Modify the plant (distribution of control surface on an aircraft or the mass and stiffness in a robot)
- Change the output (or the location of sensor)
- Change the input (or the location of actuator)