Feedback Linearization
Feedback Linearization

Consider the nonlinear system

\[ \dot{x} = f(x) + g(x)u \]
\[ y = h(x) \]

We will study the question: whether there exists a state feedback control

\[ u = \alpha(x) + \beta(x)v \]

and a change of variables

\[ z = T(x) \]

that transform the nonlinear system into an equivalent linear system.

If so, linear control design techniques can be used.
Motivation Examples

Example 1: Consider the pendulum system:

\[
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= -a \sin(x_1) - bx_2 + cu
\end{align*}
\]

Applying the control law \( u \) as

\[
u = \frac{1}{c}(a \sin(x_1) + bx_2 + v)
\]

where \( v \) is a new control, results in a linear system:

\[
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= v
\end{align*}
\]

Using pole placement method, the new control \( v \) is chosen as

\[
v = -k_1 x_1 - k_2 x_2
\]

Closed loop system is

\[
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= -k_1 x_1 - k_2 x_2
\end{align*}
\]

The overall feedback control is

\[
u = \frac{1}{c}(a \sin(x_1) + bx_2 - k_1 x_1 - k_2 x_2)
\]
Motivation

- How general is this idea of nonlinearity cancellation?
- Clearly, we should not expect to be able to cancel nonlinearities in every nonlinear system.
- There must be a certain structural property of the system that allows us to perform such cancellation.
- If the nonlinear state equation does not have the structure of such that the does this mean we cannot linearize the system via feedback? The answer is no.
- Recall that the state model of a system is not unique. It depends on the choice of the state variables.
- If the state equation does not have the structure for direct feedback cancelation for one choice of state variables, it might do so for another choice
Motivation Examples

Example 2: Consider the system

We cannot use $u$ to directly cancel $\sin(x_2)$. If we first change the variables by the transformation

$$z_1 = x_1,$$
$$z_2 = \sin(x_2)$$

Then we have

$$\dot{x}_1 = \sin(x_2) = \dot{z}_1 = z_2,$$
$$\dot{z}_2 = \cos(x_2)\dot{x}_2 = \cos(x_2)(-x_1^2 + u)$$

and the nonlinearities can be cancelled by the control

$$u = x_1^2 + \frac{1}{\cos(x_2)}v$$

which is well defined for $-\pi/2 < x_2 < \pi/2$. 
Motivation Examples

The state equation in the new coordinates \((z_1, z_2)\) can be found by inverting the transformation to express \((x_1, x_2)\) in terms of \((z_1, z_2)\); that is,

\[
    x_1 = z_1
\]

\[
    x_2 = \sin^{-1}\left(\frac{z_2}{a}\right)
\]

which is well defined for \(-a < z_2 < a\).

The transformed state equation is given by

\[
    \dot{z}_1 = z_2
\]

\[
    \dot{z}_2 = a \cos\left(\sin^{-1}\left(\frac{z_2}{a}\right)\right)(-z_1^2 + u)
\]
Transform and Diffeomorphism

When a change of variables $z = T(x)$ is used to transform the state equation from the $x$-coordinates to the $z$-coordinates, the map $T$ must be invertible; that is, it must have an inverse map $T^{-1}(\cdot)$ such that $x = T^{-1}(z)$ for all $z \in T(D_x)$, where $D_x$ is the domain of $T$.

Moreover, since the derivatives of $z$ and $z$ should be continuous, we require that both $T(\cdot)$ and $T^{-1}(\cdot)$ be continuously differentiable.

A continuously differentiable map with a continuously differentiable inverse is known as a **diffeomorphism**.
Input-State Linearizable Systems

**Definition:** A nonlinear system

\[
\dot{x} = f(x) + g(x)u
\]

where \( f : D \rightarrow \mathbb{R}^n \) and \( G : D \rightarrow \mathbb{R}^{n \times p} \) are sufficiently smooth on a domain \( D \subset \mathbb{R}^n \), is said to be input-state linearizable if there exists a diffeomorphism \( T : D \rightarrow \mathbb{R}^n \) such that \( \mathcal{D}_z = T(\mathcal{D}_x) \) contains the origin and the change of variables \( z = T(x) \) transforms the system into the form

\[
\dot{z} = Az + B\gamma(x)[u - \alpha(x)]
\]

with \((A, B)\) controllable and \( \gamma(x) \) nonsingular for all \( x \subset D \),

- **Sufficient smooth:** all the partial derivatives are defined and continuous.
Input-State Linearizable Systems

\[
\dot{z} = Az + B\gamma(x)[u-\alpha(x)]
\]

with \((A, B)\) controllable and \(\gamma(x)\) nonsingular for all \(x \subset D\),

If it is true, we can find a controller

\[
u = \alpha(x) + \beta(x)v
\]

let \(\beta(x) = \gamma^{-1}(x)\) \(v = -Kz\)

\[
\dot{z} = Az + B\gamma(x)[u-\alpha(x)] = Az - BKz = (A - BK)z
\]

where \(A-BK\) is Hurwitz.
Input-State Linearizable Systems

Example

\[ \dot{x}_1 = \sin(x_2) \]
\[ \dot{x}_2 = -x_1^2 + u \]

\[ y = x_2 \]
let \( z_1 = x_1 \), \( z_2 = a \sin x_2 \), and \( u = x_1^2 + \frac{1}{a \cos x_2} \),

It yields,
\[ \dot{z}_1 = z_2 \]
\[ \dot{z}_2 = v \]
\[ y = \sin^{-1}\left( \frac{z_2}{a} \right) \]

However, the output \( y \) is still complicated by the nonlinearity of the output equation.

Sometimes, it is more beneficial to linearize the input-output map.
Consider the single-input-single-output system
\[
\dot{x} = f(x) + g(x)u, \quad y = h(x)
\]
where \( f, g, \) and \( h \) are sufficiently smooth in a domain \( D \subset \mathbb{R}^n \). The mappings \( f : D \rightarrow \mathbb{R}^n \) and \( g : D \rightarrow \mathbb{R}^n \) are called vector fields on \( D \).

The first derivative of \( y \):
\[
\dot{y} = \frac{\partial h}{\partial x} \dot{x}
\]

\[
\dot{y} = \frac{\partial h}{\partial x} [f(x) + g(x)u] \overset{\text{def}}{=} L_f h(x) + L_g h(x) \; u
\]

\[ L_f h(x) = \frac{\partial h}{\partial x} f(x) \]

is the Lie Derivative of \( h \) with respect to \( f \) or along \( f \).

\[ L_g h(x) = \frac{\partial h}{\partial x} g(x) \]
Input-output Linearization

\[ L_g L_f h(x) = \frac{\partial (L_f h)}{\partial x} g(x) \]

\[ L_f^2 h(x) = L_f L_f h(x) = \frac{\partial (L_f h)}{\partial x} f(x) \]

\[ L_f^k h(x) = L_f L_f^{k-1} h(x) = \frac{\partial (L_f^{k-1} h)}{\partial x} f(x) \]

\[ L_f^0 h(x) = h(x) \]

\[ \dot{y} = L_f h(x) + L_g h(x) u \]

\[ L_g h(x) = 0 \quad \Rightarrow \quad \dot{y} = L_f h(x) \quad \text{independent of } u. \]

\[ y^{(2)} = \frac{\partial (L_f h)}{\partial x} [f(x) + g(x)u] = L_f^2 h(x) + L_g L_f h(x) u \]
Input-output Linearization

\[ y^{(2)} = \frac{\partial (L_fh)}{\partial x} [f(x) + g(x)u] = L_f^2 h(x) + L_g L_f h(x) \ u \]

\[ L_g L_f h(x) = 0 \ \Rightarrow \ y^{(2)} = L_f^2 h(x) \ \text{independent of } u. \]

\[ y^{(3)} = L_f^3 h(x) + L_g L_f^2 h(x) \ u \]

\[ L_g L_f^{i-1} h(x) = 0, \ i = 1, 2, \ldots, \rho - 1; \quad L_g L_f^{\rho-1} h(x) \neq 0 \]

\[ y^{(\rho)} = L_f^\rho h(x) + L_g L_f^{\rho-1} h(x) \ u \]

Now choosing

\[ u = \frac{1}{L_g L_f^{(\rho-1)} h(x)} (-L_f^{(\rho)} h(x) + v) \]

gives

\[ y^{(\rho)} = v \]

the system is input-output linearizable.
Definition: The system

\[ \dot{x} = f(x) + g(x)u, \quad y = h(x) \]

has relative degree \( \rho, 1 \leq \rho \leq n \), in \( D_0 \subset D \) if \( \forall x \in D_0 \)

\[ L_g L_f^{\rho-1} h(x) = 0, \quad i = 1, 2, \ldots, \rho - 1; \quad L_g L_f^{\rho-1} h(x) \neq 0 \]

Note: The process depends on the choice of \( y = h(x) \).
Input-output Linearization

Example

\[ \dot{x}_1 = x_2, \quad \dot{x}_2 = -x_1 + \varepsilon(1 - x_1^2)x_2 + u, \]

1) \[ y = x_1, \quad \varepsilon > 0 \]
\[ \dot{y} = \dot{x}_1 = x_2 \]
\[ \dot{y} = \dot{x}_2 = -x_1 + \varepsilon(1 - x_1^2)x_2 + u \]
Relative degree = 2 over \( R^2 \)

2) \[ y = x_2, \quad \varepsilon > 0 \]
\[ \dot{y} = \dot{x}_2 = -x_1 + \varepsilon(1 - x_1^2)x_2 + u \]
Relative degree = 1 over \( R^2 \)

3) \[ y = x_1 + x_2^2, \quad \varepsilon > 0 \]
\[ \dot{y} = x_2 + 2x_2[-x_1 + \varepsilon(1 - x_1^2)x_2 + u] \]
Relative degree = 1 over \( \{x_2 \neq 0\} \)
Example  Consider the system

\begin{align*}
\dot{x}_1 &= x_1 \\
\dot{x}_2 &= x_2 + u \\
y &= x_1
\end{align*}

Calculating the derivatives of \( y \), we obtain

\[ \dot{y} = \dot{x}_1 = x_1 = y \]

Consequently, \( y^{(n)} = y = x_1 \) for all \( n > 1 \).

In this case, the system does not have a well-defined relative degree. In this simple example, it is not difficult to see why this is so; the output

\[ y(t) = x_1(t) = e^t x_1(0) \]

is independent of the input \( u \).
Input-output Linearization

Example: Field-controlled DC motor

\[
\begin{align*}
\dot{x}_1 &= -ax_1 + u, \\
\dot{x}_2 &= -bx_2 + k - cx_1 x_3, \\
\dot{x}_3 &= \theta x_1 x_2, \quad y = x_3
\end{align*}
\]

where \( x_1, x_2, \) and \( x_3 \) are the field current, armature current, and angular velocity, respectively. \( a, b, c, k, \) and \( \theta \) are positive constants.

\[
\begin{align*}
\ddot{y} &= \dot{x}_3 = \theta x_1 x_2 \\
\ddot{y} &= \theta x_1 \dot{x}_2 + \theta \dot{x}_1 x_2 = (\cdot) + \theta x_2 u
\end{align*}
\]

Relative degree = 2 over \( \{ x_2 \neq 0 \} \)
Normal Form

Change of variables:

\[
z = T(x) = \begin{bmatrix}
\phi_1(x) \\
\vdots \\
\phi_{n-\rho}(x) \\
- - - \\
h(x) \\
\vdots \\
L_f^{\rho-1}h(x)
\end{bmatrix}
\]

are chosen such that \( T(x) \) is a diffeomorphism on a domain \( D_0 \subset D \)
Normal Form

\[
\dot{\eta} = \frac{\partial \phi}{\partial x}[f(x) + g(x)u] = f_0(\eta, \xi) + g_0(\eta, \xi)u
\]

\[
\dot{\xi}_i = \xi_{i+1}, \quad 1 \leq i \leq \rho - 1
\]

\[
\dot{\xi}_\rho = L_f^\rho h(x) + L_g L_f^{\rho-1} h(x)\ u
\]

\[
y = \xi_1
\]

Choose $\phi(x)$ such that $T(x)$ is a diffeomorphism and

\[
\frac{\partial \phi_i}{\partial x} g(x) = 0, \quad \text{for } 1 \leq i \leq n - \rho, \quad \forall \ x \in D_0
\]

Always possible (at least locally)

\[
\dot{\eta} = f_0(\eta, \xi)
\]
Normal Form

**Theorem:** Suppose the system

\[
\dot{x} = f(x) + g(x)u, \quad y = h(x)
\]

has relative degree \( \rho (\leq n) \) in \( D \). If \( \rho = n \), then for every \( x_0 \subset D \), a neighborhood \( N \) of \( x_0 \) exists such that the map \( T(x) = \psi(x) \), restricted to \( N \), is a diffeomorphism on \( N \). If \( \rho < n \), then, for every \( x_0 \subset D \), a neighborhood \( N \) of \( x_0 \) and smooth functions \( \varphi_1(x), \ldots, \varphi_{n-\rho}(x) \) exist such that

\[
\frac{\partial \phi_i}{\partial x} g(x) = 0, \quad \text{for } 1 \leq i \leq n - \rho
\]

is satisfied for all \( x \in N \) and the map \( T(x) = \begin{bmatrix} \phi(x) \\ \psi(x) \end{bmatrix} \), restricted to \( N \), is a diffeomorphism on \( N \).
Normal Form:

\[
\begin{align*}
\dot{\eta} &= f_0(\eta, \xi) \\
\dot{\xi}_i &= \xi_{i+1}, \quad 1 \leq i \leq \rho - 1 \\
\dot{\xi}_\rho &= L_f^\rho h(x) + L_g L_f^{\rho-1} h(x) \ u \\
y &= \xi_1
\end{align*}
\]

\[
A_c = \begin{bmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & 0 & 1 & \vdots \\
0 & \ldots & \ldots & 0 & 0
\end{bmatrix}, \quad B_c = \begin{bmatrix}
0 \\
0 \\
\vdots \\
0 \\
1
\end{bmatrix}
\]

\[
C_c = \begin{bmatrix}
1 & 0 & \ldots & 0 & 0
\end{bmatrix}
\]
Normal Form:

\[
\begin{align*}
\dot{\eta} &= f_0(\eta, \xi) \\
\dot{\xi} &= A_c \xi + B_c \left[ L_f^\rho h(x) + L_g L_f^\rho L_f^{-1} h(x) u \right] \\
y &= C_c \xi \\
\dot{\eta} &= \frac{\partial \phi}{\partial x} f(x) \bigg|_{x = T^{-1}(z)} = f_0(\eta, \xi) \\
\gamma(x) &= L_g L_f^\rho L_f^{-1} h(x), \quad \alpha(x) = - \frac{L_f^\rho h(x)}{L_g L_f^\rho L_f^{-1} h(x)} \\
\dot{\xi} &= A_c \xi + B_c \gamma(x)[u - \alpha(x)]
\end{align*}
\]

If \( x^* \) is an open-loop equilibrium point at which \( y = 0 \); i.e., \( f(x^*) = 0 \) and \( h(x^*) = 0 \), then \( \psi(x^*) = 0 \). Take \( \phi(x^*) = 0 \) so that \( z = 0 \) is an open-loop equilibrium point.
Zero Dynamics

Normal Form:

\[
\begin{align*}
\dot{\eta} &= f_0(\eta, \xi) \\
\dot{\xi} &= A_c\xi + B_c\gamma(x)[u - \alpha(x)] \\
y &= C_c\xi
\end{align*}
\]

\[y(t) \equiv 0\]

\[y(t) \equiv 0 \Rightarrow \xi(t) \equiv 0 \Rightarrow u(t) \equiv \alpha(x(t)) \Rightarrow \dot{\eta} = f_0(\eta, 0)\]

**Definition:** The equation \( \dot{\eta} = f_0(\eta, 0) \) is called the zero dynamics of the system. The system is said to be minimum phase if zero dynamics have an asymptotically stable equilibrium point in the domain of interest (at the origin if \( T(0) = 0 \)).
Zero Dynamics

\[ \dot{\eta} = f_0(\eta, 0) \]

The zero dynamics can be characterized in the \( x \)-coordinates

\[ y(t) \equiv 0 \Rightarrow \xi(t) \equiv 0 \Rightarrow u(t) \equiv \alpha(x(t)) \Rightarrow \dot{\eta} = f_0(\eta, 0) \]

\[ Z^* = \{ x \in D_0 \mid h(x) = L_f h(x) = \cdots = L_f^{\rho-1} h(x) = 0 \} \]

\[ y(t) \equiv 0 \Rightarrow x(t) \in Z^* \]

\[ \Rightarrow u = u^*(x) \overset{\text{def}}{=} \alpha(x)\big|_{x \in Z^*} \]

The restricted motion of the system is described by

\[ \dot{x} = f^*(x) \overset{\text{def}}{=} [f(x) + g(x)\alpha(x)]_{x \in Z^*} \]
In the special case $\rho = n$, the norm form reduces

$$
\dot{z} = A_c z + B_c \gamma(x)[u - \alpha(x)] \\
y = C_c z
$$

where $z = \zeta = [h(x), \ldots, L_{f}^{n-1}h(x)]^T$ and the $\eta$ variable does not exist.

In this case, the system has no zero dynamics and by default, is said to be minimum phase.
Example

Consider the controlled van der Pol equation

\[
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= -x_1 + \varepsilon(1 - x_1^2)x_2 + u, \\
y &= x_2
\end{align*}
\]

\[\Rightarrow \quad \rho = 1\]

Taking \(\zeta = y\) and \(\eta = x_1\), we see that the system is already in the normal form

\[y(t) \equiv 0 \quad \Rightarrow \quad x_2(t) \equiv 0 \quad \Rightarrow \quad \dot{x}_1 = 0\]

The zero dynamics does not have an asymptotically stable equilibrium point.

Non-minimum phase
Example Consider the system
\[
\begin{align*}
\dot{x}_1 &= -x_1 + \frac{2 + x_3^2}{1 + x_3^2} u, \\
\dot{x}_2 &= x_3, \\
\dot{x}_3 &= x_1 x_3 + u, \\
y &= x_2
\end{align*}
\]
\[
\begin{align*}
\ddot{y} &= \dot{x}_2 = x_3 \\
\dot{y} &= \dot{x}_3 = x_1 x_3 + u \quad \Rightarrow \quad \rho = 2 \\
\gamma &= L_g L_f h(x) = 1, \quad \alpha = -\frac{L_f^2 h(x)}{L_g L_f h(x)} = -x_1 x_3 \\
Z^* &= \{x_2 = x_3 = 0\} \\
\Rightarrow \quad u &= u^*(x) \overset{\text{def}}{=} \alpha(x)|_{x \in Z^*}
\end{align*}
\]
\[
\begin{align*}
u &= u^*(x) = 0 \quad \Rightarrow \quad \dot{x}_1 &= -x_1
\end{align*}
\]
Minimum phase
Example

To transform the system into the normal form, we want to choose a function \( \phi(x) \) such that

\[
\phi(0) = 0, \quad \frac{\partial \phi}{\partial x} g(x) = \begin{bmatrix}
\frac{\partial \phi}{\partial x_1}, & \frac{\partial \phi}{\partial x_2}, & \frac{\partial \phi}{\partial x_3}
\end{bmatrix}
\begin{bmatrix}
\frac{2+x_3^2}{1+x_3^2} \\
0 \\
1
\end{bmatrix} = 0
\]

and

\[
T(x) = \begin{bmatrix}
\phi(x) & x_2 & x_3
\end{bmatrix}^T
\]

is a diffeomorphism. The partial differential equation

\[
\frac{\partial \phi}{\partial x_1} \cdot \frac{2 + x_3^2}{1 + x_3^2} + \frac{\partial \phi}{\partial x_3} = 0
\]

can be solved by separation of variables to obtain

\[
\phi(x) = -x_1 + x_3 + \tan^{-1} x_3
\]

which satisfies the condition \( \phi(0) = 0 \).
Example

\[ \frac{\partial \phi}{\partial x_1} \cdot \frac{2 + x_3^2}{1 + x_3^2} + \frac{\partial \phi}{\partial x_3} = 0 \]

It is a linear and homogeneous PDE for the unknown \( \phi \).

Solution: Integrate

\[ dx_3 \frac{2 + x_3^2}{1 + x_3^2} = dx_3(1 + \frac{1}{1 + x_3^2}) = -dx_1 \quad \text{and} \quad d\phi = 0 \]

to obtain

\[ -x_1 + x_3 + \tan^{-1}x_3 = c_1 \quad \text{and} \quad \phi(x) = c_2 \]

Choosing \( \phi(x) = \phi(-x_1 + x_3 + \tan^{-1}x_3) = -x_1 + x_3 + \tan^{-1}x_3 \) ensure that \( \phi(0) = 0 \)
Example

\[ T(x) = \begin{bmatrix} -x_1 + x_3 + \tan^{-1} x_3, & x_2, & x_3 \end{bmatrix}^T \]

The mapping \( T(x) \) is a global diffeomorphism, as can be seen by the fact that for any \( z \in \mathbb{R}^3 \), the equation \( T(x) = z \) has a unique solution. Thus, the normal form

\[
\begin{align*}
\eta &= -x_1 + x_3 + \tan^{-1} x_3, \quad \xi_1 = x_2, \quad \xi_2 = x_3 \\
\dot{\eta} &= -\dot{x}_1 + \dot{x}_3 + \frac{\dot{x}_3}{1+x_3^2} = -\dot{x}_1 + (\frac{2+x_3^2}{1+x_3^2})\dot{x}_3 = x_1(1+\frac{2+x_3^2}{1+x_3^2} x_3) \\
x_1 &= -\eta + x_3 + \tan^{-1} x_3 = -\eta + \xi_2 + \tan^{-1} \xi_2 \\
\dot{\eta} &= (-\eta + \xi_2 + \tan^{-1} \xi_2) \left(1 + \frac{2 + \xi_2^2}{1 + \xi_2^2} \right) \\
\dot{\xi}_1 &= \xi_2 \\
\dot{\xi}_2 &= (-\eta + \xi_2 + \tan^{-1} \xi_2) \xi_2 + u \\
y &= \xi_1
\end{align*}
\]

is defined globally.
Full-State Linearization

Definition: A nonlinear system is in the controller form if

\[ \dot{x} = Ax + B\gamma(x)[u - \alpha(x)] \]

where \((A,B)\) is controllable and \(\gamma(x)\) is a nonsingular

\[ u = \alpha(x) + \gamma^{-1}(x)v \]

\[ \Downarrow \]

\[ \dot{x} = Ax + Bv \]
Full-State Linearization

The $n$-dimensional single-input (SI) system

$$\dot{x} = f(x) + g(x)u$$

can be transformed into the controller form if $\exists \ h(x)$ s.t.

$$\dot{x} = f(x) + g(x)u, \quad y = h(x)$$

has relative degree $n$. Furthermore, the normal form reduces to

$$\dot{z} = A_c z + B_c \gamma(z)[u - \alpha(z)],$$
$$y = C_c z$$
Full-State Linearization

On the other hand, if there is a change of variables $\zeta = S(x)$ that transforms the SI system

$$\dot{x} = f(x) + g(x)u$$

into the controller form

$$\dot{\zeta} = A\zeta + B\gamma(\zeta)[u - \alpha(\zeta)]$$

then there is a function $h(x)$ such that the system

$$\dot{x} = f(x) + g(x)u, \quad y = h(x)$$

has relative degree $n$. 
Full-State Linearization

For any controllable pair \((A,B)\), we can find a nonsingular matrix \(M\) that transforms \((A,B)\) into a controllable canonical form:

\[
M A M^{-1} = A_c + B_c \lambda^T, \\
M B = B_c \\
z = M \zeta = MS(x) \overset{\text{def}}{=} T(x) \\
\dot{z} = A_c z + B_c \gamma(\cdot)[u - \alpha(\cdot)] \\
h(x) = T_1(x)
\]
Full-State Linearization

In summary, the $n$-dimensional SI system
\[ \dot{x} = f(x) + g(x)u \]
is transformable into the controller form if and only if $\exists \ h(x)$ such that
\[ \dot{x} = f(x) + g(x)u, \quad y = h(x) \]
has relative degree $n$.

Search for a smooth function $h(x)$ such that
\[ L_g L_f^{i-1} h(x) = 0, \quad i = 1, 2, \ldots, n-1, \quad \text{and} \quad L_g L_f^{n-1} h(x) \neq 0 \]
\[ T(x) = \begin{bmatrix} h(x), \quad L_f h(x), \quad \cdots \quad L_f^{n-1} h(x) \end{bmatrix}^T \]

These conditions use the notions of Lie brackets and invariant distributions, which we introduce next.
Full-State Linearization

The Lie Bracket: For two vector fields $f$ and $g$, the Lie bracket $[f, g]$ is a third vector field defined by

$$[f, g](x) = \frac{\partial g}{\partial x} f(x) - \frac{\partial f}{\partial x} g(x)$$

The following notations is used to simplify this process:

$$ad^0_f g(x) = g(x), \quad ad_f g(x) = [f, g](x)$$

$$ad^k_f g(x) = [f, ad^{k-1}_f g](x), \quad k \geq 1$$

Properties:

- $[f, g] = -[g, f]$

- For constant vector fields $f$ and $g$, $[f, g] = 0$
Full-State Linearization

Example

\[
\begin{align*}
f &= \begin{bmatrix}
x_2 \\
-\sin x_1 - x_2
\end{bmatrix}, \quad g &= \begin{bmatrix}
0 \\
x_1
\end{bmatrix} \\
[f, g](x) &= \frac{\partial g}{\partial x} f(x) - \frac{\partial f}{\partial x} g(x)
\end{align*}
\]

\[
[f, g] = \begin{bmatrix}
0 & 0 \\
1 & 0
\end{bmatrix} \begin{bmatrix}
x_2 \\
-\sin x_1 - x_2
\end{bmatrix} - \begin{bmatrix}
0 & 1 \\
-\cos x_1 & -1
\end{bmatrix} \begin{bmatrix}
0 \\
x_1
\end{bmatrix}
\]

\[
= \begin{bmatrix}
-x_1 \\
x_1 + x_2
\end{bmatrix} \overset{\text{def}}{=} ad_f g = [f, g]
\]

Full-State Linearization

Example (continue)

\[
f = \begin{bmatrix} x_2 \\ -\sin x_1 - x_2 \end{bmatrix}, \quad ad_f g = \begin{bmatrix} -x_1 \\ x_1 + x_2 \end{bmatrix}
\]

\[
ad_f^2 g = [f, ad_f g] = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ -\sin x_1 - x_2 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -\cos x_1 & -1 \end{bmatrix} \begin{bmatrix} -x_1 \\ x_1 + x_2 \end{bmatrix} = \begin{bmatrix} -x_1 - 2x_2 \\ x_1 + x_2 - \sin x_1 - x_1 \cos x_1 \end{bmatrix}
\]
**Full-State Linearization**

**Distribution:** For vector fields \( f_1, f_2, \ldots, f_k \) on \( D \subset \mathbb{R}^n \), let

\[
\Delta(x) = \text{span}\{f_1(x), f_2(x), \ldots, f_k(x)\}
\]

The collection of all vector spaces \( \Delta(x) \) for \( x \in D \) is called a **distribution** and referred to by \( \Delta = \text{span}\{f_1, f_2, \ldots, f_k\} \)

If \( \dim(\Delta(x)) = k \) for all \( x \in D \), we say that \( \Delta \) is a nonsingular distribution on \( D \), generated by \( f_1, \ldots, f_k \)

A distribution \( \Delta \) is **involutive** if

\[
g_1 \in \Delta \text{ and } g_2 \in \Delta \Rightarrow [g_1, g_2] \in \Delta
\]

**Lemma:** If \( \Delta \) is a nonsingular distribution, generated by \( f_1, \ldots, f_k \), then it is **involutive** if and only if

\[
[f_i, f_j] \in \Delta, \quad \forall 1 \leq i, j \leq k
\]
Full-State Linearization

Example: \( D = \mathbb{R}^3; \Delta = \text{span}\{f_1, f_2\} \)

\[
f_1 = \begin{bmatrix} 2x_2 \\ 1 \\ 0 \end{bmatrix}, \quad f_2 = \begin{bmatrix} 1 \\ 0 \\ x_2 \end{bmatrix}, \quad \dim(\Delta(x)) = 2, \quad \forall x \in D
\]

\[
[f_1, f_2] = \frac{\partial f_2}{\partial x} f_1 - \frac{\partial f_1}{\partial x} f_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}
\]

\[
\text{rank } [f_1(x), f_2(x), [f_1, f_2](x)] = \begin{bmatrix} 2x_2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & x_2 & 1 \end{bmatrix}
\]

\[
\text{rank } \begin{bmatrix} 2x_2 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & x_2 & 1 \end{bmatrix} = 3, \quad \forall x \in D
\]

\( \Delta \) is not involutive
Full-State Linearization

Example: \( D = \{ x \in \mathbb{R}^3 \mid x_1^2 + x_3^2 \neq 0 \}; \Delta = \text{span}\{f_1, f_2\} \)

\[
\begin{bmatrix}
2x_3 \\
-1 \\
0
\end{bmatrix}, \begin{bmatrix}
-x_1 \\
-2x_2 \\
x_3
\end{bmatrix}, \quad \text{dim}(\Delta(x)) = 2, \forall x \in D
\]

\[
[f_1, f_2] = \frac{\partial f_2}{\partial x} f_1 - \frac{\partial f_1}{\partial x} f_2 = \begin{bmatrix}
-4x_3 \\
2 \\
0
\end{bmatrix}
\]

\[
\text{rank} [f_1(x), f_2(x), [f_1, f_2](x)] = \begin{bmatrix}
2x_3 & -x_1 & -4x_3 \\
-1 & -2x_2 & 2 \\
0 & x_3 & 0
\end{bmatrix}
\]

\[
\text{rank} = 2, \forall x \in D
\]

\( \Delta \) is involutive
Full-State Linearization

**Theorem:** The $n$-dimensional SI system

$$\dot{x} = f(x) + g(x)u$$

is transformable into the controller form if and only if there is a domain $D_0$ such that

$$\text{rank}[g(x), ad_f g(x), \ldots, ad_f^{n-1} g(x)] = n, \ \forall \ x \in D_0$$

and

$$\text{span} \ \{g, ad_f g, \ldots, ad_f^{n-2} g\} \ \text{is involutive in} \ D_0$$

**Proof:** See Appendix C:22 (Khalil’s Book)
Full-State Linearization

Example

\[
\dot{x} = \begin{bmatrix}
  a \sin x_2 \\
  -x_1^2
\end{bmatrix} + \begin{bmatrix}
  0 \\
  1
\end{bmatrix} u
\]

\[
ad_fg = [f, g] = - \frac{\partial f}{\partial x} g = \begin{bmatrix}
  -a \cos x_2 \\
  0
\end{bmatrix}
\]

\[
[g(x), ad_fg(x)] = \begin{bmatrix}
  0 & -a \cos x_2 \\
  1 & 0
\end{bmatrix}
\]

\[
\text{rank}[g(x), ad_fg(x)] = 2, \, \forall \, x \text{ such that } \cos x_2 \neq 0
\]

\[
\text{span}\{g\} \text{ is involutive}
\]

Find \( h \) such that \( L_g h(x) = 0 \), and \( L_g L_f h(x) \neq 0 \)
Full-State Linearization

\[
\frac{\partial h}{\partial x} g = \begin{bmatrix} \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{\partial h}{\partial x_2} = 0 \quad \Rightarrow \quad h \text{ is independent of } x_2
\]

\[
L_f h(x) = \frac{\partial h}{\partial x} f = \begin{bmatrix} \frac{\partial h}{\partial x_1} & 0 \end{bmatrix} \begin{bmatrix} a \sin x_2 \\ 1 \end{bmatrix} = \frac{\partial h}{\partial x_1} a \sin x_2
\]

\[
L_g L_f h(x) = \frac{\partial (L_f h)}{\partial x} g = \frac{\partial (L_f h)}{\partial x_2} = \frac{\partial h}{\partial x_1} a \cos x_2 \neq 0
\]

\[
L_g L_f h(x) \neq 0 \text{ in } D_0 = \{ x \in \mathbb{R}^2 | \cos x_2 \neq 0 \} \text{ if } \frac{\partial h}{\partial x_1} \neq 0
\]

Take \( h(x) = x_1 \quad \Rightarrow \quad T(x) = \begin{bmatrix} h \\ L_f h \end{bmatrix} = \begin{bmatrix} x_1 \\ a \sin x_2 \end{bmatrix} \)
Full-State Linearization

Example (Field-Controlled DC Motor) \[ \dot{x} = f(x) + g(x)u \]

\[ \dot{x} = \begin{bmatrix} -ax_1 \\ -bx_2 + k - cx_1x_3 \\ \theta x_1x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u \]

where \( x_1, x_2, \) and \( x_3 \) are the field current, armature current, and angular velocity, respectively.

\[ ad_f g = \begin{bmatrix} a \\ cx_3 \\ -\theta x_2 \end{bmatrix}; \quad ad_f^2 g = \begin{bmatrix} a^2 \\ (a + b)cx_3 \\ (b - a)\theta x_2 - \theta k \end{bmatrix} \quad ad_f^2 g = [f, ad_f g] \]

\[ [g(x), ad_f g(x), ad_f^2 g(x)] = \begin{bmatrix} 1 & a & a^2 \\ 0 & cx_3 & (a + b)cx_3 \\ 0 & -\theta x_2 & (b - a)\theta x_2 - \theta k \end{bmatrix} \]
Full-State Linearization

\[
\dot{x} = \begin{bmatrix}
-a x_1 \\
d d x_2 + k - c x_1 x_3 \\
\theta x_1 x_2 \\
\end{bmatrix} + \begin{bmatrix}
1 \\
0 \\
0 \\
\end{bmatrix} u
\]

\[
ad_f g = [f, g] = \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g = -\frac{\partial f}{\partial x} g = -\left[\frac{\partial f}{\partial x_1} \frac{\partial f}{\partial x_2} \frac{\partial f}{\partial x_3}\right]^T \begin{bmatrix}
1 \\
0 \\
0 \\
\end{bmatrix} = -\left[\frac{\partial f}{\partial x_1}\right] = \begin{bmatrix}
a \\
c x_3 \\
-\theta x_2 \\
\end{bmatrix}
\]

\[
ad_f^g = [f, ad_f g] = \frac{\partial ad_f g}{\partial x} f - \frac{\partial f}{\partial x} ad_f g = \begin{bmatrix}
0 & 0 & 0 \\
0 & c & -b x_2 + k - c x_3 \\
0 & -\theta & 0 \\
\end{bmatrix} - \begin{bmatrix}
0 \\
d d x_1 x_2 \\
\theta x_1 x_2 \\
\end{bmatrix} \begin{bmatrix}
-a \\
0 \\
0 \\
\end{bmatrix} = \begin{bmatrix}
-a^2 \\
- c x_3 - b c x_3 + c \theta x_1 x_3 \\
a \theta x_2 + c \theta x_1 x_3 \\
\end{bmatrix} = \begin{bmatrix}
a^2 \\
c(a + b) x_3 \\
(b - a) \theta x_2 - \theta k \\
\end{bmatrix}
\]
Full-State Linearization

\[ \det[\cdot] = c\theta(-k + 2bx_2)x_3 \]

\(a, b, c, k,\) and \(\theta\) are positive constants.

\[ \text{rank } [\cdot] = 3 \text{ for } x_2 \neq k/2b \text{ and } x_3 \neq 0 \]

\(\text{span}\{g, ad_fg\}\) is involutive if \([g, ad_fg] \in \text{span}\{g, ad_fg\}\)

\[
[g, ad_fg] = \frac{\partial(ad_fg)}{\partial x}g = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & c \\ 0 & -\theta & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\]

\[ \Rightarrow \text{span}\{g, ad_fg\}\) is involutive

\[ D_0 = \{x \in R^3 | x_2 > \frac{k}{2b} \text{ and } x_3 > 0\} \]

Find \(h\) such that \(L_g h(x) = L_g L_f h(x) = 0; \ L_g L_f^2 h(x) \neq 0\)
Full-State Linearization

\[ x^* = [0, k/b, \omega_0]^T, \quad h(x^*) = 0 \]

\[ \frac{\partial h}{\partial x} g = \begin{bmatrix} \frac{\partial h}{\partial x_1} & \frac{\partial h}{\partial x_2} & \frac{\partial h}{\partial x_3} \\ 1 & 0 & 0 \end{bmatrix} = \frac{\partial h}{\partial x_1} = 0 \Rightarrow h \text{ is independent of } x_1 \]

\[ L_f h(x) = \frac{\partial h}{\partial x_2} [-bx_2 + k - cx_1 x_3] + \frac{\partial h}{\partial x_3} \theta x_1 x_2 \]

\[ [\partial (L_f h)/\partial x] g = 0 \Rightarrow cx_3 \frac{\partial h}{\partial x_2} = \theta x_2 \frac{\partial h}{\partial x_3} \]

\[ h = c_1 [\theta x_2^2 + c x_3^2] + c_2, \quad L_g L_f^2 h(x) = -2c_1 c \theta (k - 2bx_2) x_3 \]

\[ h(x^*) = c_1 [\theta (k/b)^2 + c \omega_0^2] + c_2 \]

\[ c_1 = 1, \quad c_2 = -\theta (k/b)^2 - c \omega_0^2 \]
State Feedback Control

Consider a partial feedback linearizable system

\[ \dot{\eta} = f_0(\eta, \xi) \]
\[ \dot{\xi} = A\xi + B\gamma(x)(u - \alpha(x)) \]
\[ y = C\xi \]

where

\[ z = \begin{bmatrix} \eta \\ \xi \end{bmatrix} = T(x) = \begin{bmatrix} T_1(x) \\ T_2(x) \end{bmatrix} \]

\( T(x) \) is a diffeomorphism on a domain \( D \subset \mathbb{R}^n \), \( D_z = T(D) \)

contains the origin, \((A,B)\) is a controllable pair, \( \gamma(x) \)
is nonsingular for all \( x \in D \), \( f_0(0,0) = 0 \), \( f_0(\eta, \xi), \alpha(x), \gamma(x) \)
are continuously differentiable.

Goal: Design the feedback \( u \) to stabilize \( z \) at the origin.
State Feedback Control

The state feedback

\[ u = \alpha(x) - \gamma^{-1}(x)K\xi \]  \hspace{1cm} (a)

results in closed loop

\[ \dot{\eta} = f_0(\eta, \xi) \]
\[ \dot{\xi} = (A - BK)\xi \]  \hspace{1cm} (b)

**Lemma 1.** The origin of (b) is asymptotically stable if the origin of \( \dot{\eta} = f_0(\eta, 0) \) is asymptotically stable and the feedback gain \( K \) is chosen such that \( A - BK \) is Hurwitz.
Proof: Since $\dot{\eta} = f_0(\eta, 0)$ is asymptotically stable, there exists a Lyapunov function $V_1(\eta)$ such that

$$\frac{\partial V_1(\eta)}{\partial \eta} f_0(\eta, 0) \leq -\alpha_3(\|\eta\|)$$

- Take the Lyapunov function $V = V_1 + k\sqrt{\xi^T P \xi}$ with $P$ being a solution of $P(A - BK) + (A - BK)^T P = -I$.

- The first time derivative of $V$ satisfies:

$$\dot{V} \leq -\alpha_3(\|\eta\|) - k_1 \|\xi\|$$

implies that the origin is asymptotically stable.
State Feedback Control

Example

\[ \dot{\eta} = -\eta + \eta^2 \xi, \quad \dot{\xi} = v \]

The origin of \( \dot{\eta} = -\eta \) is globally exponentially stable, but the origin of

\[ \dot{\eta} = -\eta + \eta^2 \xi, \quad \dot{\xi} = -k\xi, \quad k > 0 \]

is not globally asymptotically stable.

The region of attraction is \( \{ \eta \xi < 1 + k \} \)
State Feedback Control

The state feedback

\[ u = \alpha(x) - \gamma^{-1}(x)K\xi \]  

(a)

results in closed loop

\[ \dot{\eta} = f_0(\eta, \xi) \]

\[ \xi = (A - BK)\xi \]  

(b)

Lemma 2. The origin of (b) is asymptotically stable if the origin of \( \dot{\eta} = f_0(\eta, 0) \) is input-to-state stable and the feedback gain \( K \) is chosen such that \( A - BK \) is Hurwitz.
Consider the nonlinear system

\[ \dot{x} = f(x) + G(x)u \]

\[ f(0) = 0, \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m \]

Suppose there is a change of variables \( z = T(x) \), defined
for all \( x \in D \subset \mathbb{R}^n \), that transforms the system into the controller form

\[ \dot{z} = Az + B\gamma(x)[u - \alpha(x)] \]

where \((A,B)\) is controllable and \( \gamma(x) \) is nonsingular for all \( x \in D \)

\[ u = \alpha(x) + \gamma^{-1}(x)v \quad \Rightarrow \quad \dot{z} = Az + Bv \]
Robust Issue of State Feedback Control

\[ u = \alpha(x) + \gamma^{-1}(x)v \Rightarrow \dot{z} = Az + Bv \]
\[ v = -Kz \]

Design \( K \) such that \((A - BK)\) is Hurwitz

The origin \( z = 0 \) of the closed-loop system

\[ \dot{z} = (A - BK)z \]

is globally exponentially stable, with the feedback control law:

\[ u = \alpha(x) - \gamma^{-1}(x)KT(x). \]

Closed-loop system in the \( x \)-coordinates:

\[ \dot{x} = f(x) + G(x) [\alpha(x) - \gamma^{-1}(x)KT(x)] \]
Robust Issue of State Feedback Control

- This beautiful result state feedback linearization, however, is based on exact mathematical cancellation of the nonlinear terms: $\alpha$, $\gamma$, and $T$.

- Exact cancellation is almost impossible for several practical reasons such as model simplification, parameter uncertainty, and computational errors.

  What is the effect of uncertainty in $\alpha$, $\gamma$, and $T$?
Robust Issue of State Feedback Control

- Most likely, the controller will be implementing functions \( \hat{\alpha}(x), \hat{\gamma}(x), \) and \( \hat{T}(x) \) which are approximations of \( \alpha, \gamma, \) and \( T; \) that is to say, the actual controller will be implementing the feedback control law

\[
u = \hat{\alpha}(x) - \hat{\gamma}^{-1}(x) K \hat{T}(x)
\]

Closed-loop system:

\[
\dot{z} = (A - BK)z + B\delta(z)
\]

\[
\delta = \gamma[\hat{\alpha} - \alpha + \gamma^{-1}KT - \hat{\gamma}^{-1}K\hat{T}]
\]

Thus, the closed-loop system appears as a perturbation of the nominal system

\[
\dot{z} = (A - BK)z
\]
Robust Issue of State Feedback Control

\[ \dot{z} = (A - BK)z + B\delta(z) \quad (*) \]

**Lemma 3** Consider the closed-loop system (*), where \((A - BK)\) is Hurwitz. Let

\[ V(z) = z^T P z, \quad P(A - BK) + (A - BK)^T P = -I \]

and

- If \(\|\delta(z)\| \leq k \|z\|\) for all \(z\), where

\[ 0 \leq k < \frac{1}{2\|PB\|} \]

then the origin of (*) is globally exponentially stable

- If \(\|\delta(z)\| \leq k \|z\| + \varepsilon\) for all \(z\), then the state \(z\) is globally ultimately bounded by \(\varepsilon c\) for some \(c > 0\)
Robust Issue of State Feedback Control

Example (Pendulum Equation):

$$\ddot{\theta} = -a \sin \theta - b \dot{\theta} + cT$$

$$x_1 = \theta - \delta, \quad x_2 = \dot{\theta}, \quad u = T - T_{ss} = T - \frac{a}{c} \sin \delta$$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -a \sin(x_1 + \delta) - b x_2 + c u$$

$$u = \frac{1}{c} \left\{ a \left[ \sin(x_1 + \delta) - \sin \delta \right] - k_1 x_1 - k_2 x_2 \right\}$$

$$A - BK = \begin{bmatrix} 0 & 1 \\ -k_1 & -(k_2 + b) \end{bmatrix} \text{ is Hurwitz}$$
Robust Issue of State Feedback Control

\[ T = u + \frac{a}{c} \sin \delta = \frac{1}{c} [a \sin(x_1 + \delta) - k_1 x_1 - k_2 x_2] \]

Let \( \hat{a} \) and \( \hat{c} \) be nominal models of \( a \) and \( c \)

\[ T = \frac{1}{\hat{c}} [\hat{a} \sin(x_1 + \delta) - k_1 x_1 - k_2 x_2] \]

\[ \dot{x} = (A - BK)x + B\delta(x) \]

\[ \delta(x) = \left( \frac{\hat{a}c - ac}{\hat{c}} \right) \sin(x_1 + \delta_1) - \left( \frac{c - \hat{c}}{\hat{c}} \right) (k_1 x_1 + k_2 x_2) \]
Robust Issue of State Feedback Control

\[
\delta(x) = \left(\frac{\hat{a}c - a\hat{c}}{\hat{c}}\right) \sin(x_1 + \delta_1) - \left(\frac{c - \hat{c}}{\hat{c}}\right) (k_1 x_1 + k_2 x_2)
\]

\[
|\delta(x)| \leq k\|x\| + \varepsilon
\]

\[
k = \left|\frac{\hat{a}c - a\hat{c}}{\hat{c}}\right| + \left|\frac{c - \hat{c}}{\hat{c}}\right| \sqrt{k_1^2 + k_2^2}, \quad \varepsilon = \left|\frac{\hat{a}c - a\hat{c}}{\hat{c}}\right| \sin \delta_1
\]

\[
P = \begin{bmatrix}
p_{11} & p_{12} \\
p_{12} & p_{22}
\end{bmatrix}, \quad PB = \begin{bmatrix}
p_{12} \\
p_{22}
\end{bmatrix}
\]

\[
k < \frac{1}{2\sqrt{p_{12}^2 + p_{22}^2}}
\]

\[
\sin \delta_1 = 0 \quad \Rightarrow \quad \varepsilon = 0
\]
Robust Issue of State Feedback Control

Lemma 4

1. If \( \| \delta(z) \| \leq \varepsilon \) for all \( z \) and \( \dot{\eta} = f_0(\eta, \xi) \) is input-to-state stable, then the state \( z \) is globally ultimately bounded by a class \( K \) function of \( \varepsilon \).

2. If \( \| \delta(z) \| \leq k \| z \| \) in some neighborhood of \( z = 0 \), with sufficiently small \( k \), and the origin of \( \dot{\eta} = f_0(\eta, 0) \) is exponentially stable, then \( z = 0 \) is an exponentially stable equilibrium point of the system

\[
\dot{\eta} = f_0(\eta, \xi), \quad \dot{\xi} = (A - BK)\xi + B\delta(z)
\]
Efficiency of State Feedback Control

• Is feedback linearization a good idea?

Example  Consider the scalar system

\[ \dot{x} = ax - bx^3 + u, \quad a, b > 0 \]
\[ u = -(k + a)x + bx^3, \quad k > 0, \quad \Rightarrow \quad \dot{x} = -kx \]

\(-bx^3\) is a damping term. Why cancel it?

\[ u = -(k + a)x, \quad k > 0, \quad \Rightarrow \quad \dot{x} = -kx - bx^3 \]

Which design is better?

\[ \dot{x} = -kx - bx^3 \]

whose origin is globally asymptotically stable and its trajectories approach the origin faster than \( \dot{x} = -kx \).
Robust Issue of State Feedback Control

- The robustness and efficiency concerns we raised regarding feedback linearization as a design procedure should not undermined the feedback linearization theory we developed in this chapter.

- The theory provides us with valuable tools to characterize a class of nonlinear systems whose structure is open to feedback control design, with or without nonlinearity cancellation.

- The concepts of relative degree and zero dynamics of nonlinear systems play a crucial role in the feedback design procedure of nonlinear systems.

- The ability to transform a system into a normal form brings in the matching condition structure that will be used in develop some useful robust control techniques for the control of nonlinear systems.