P4.12. Given the original canonical form of the performance surface

$$\xi = \xi_{min} + \mathbf{v}^H \mathbf{R} \mathbf{v}$$

$\mathbf{v} = \mathbf{w} - \mathbf{w}_o$ is the tap weight error vector, and $\mathbf{R}$ is the autocorrelation matrix which can be written in its diagonalized form

$$\begin{aligned} \xi &= \xi_{min} + \mathbf{v}^H \mathbf{Q}\mathbf{\Lambda}\mathbf{Q}^H \mathbf{v} \\ &= \xi_{min} + (\mathbf{Q}^H\mathbf{v})^H \mathbf{\Lambda}(\mathbf{Q}^H\mathbf{v}) \\ &= \xi_{min} + \mathbf{v}'^H \mathbf{\Lambda}\mathbf{v}' \end{aligned}$$

P4.13. (i)

$$x(n) = a_1\, e^{j\omega_1 n} + a_2\, e^{j\omega_2 n} + \nu(n)$$

$$x(n)x^*(n-k) = (a_1 e^{j\omega_1 n} + a_2 e^{j\omega_2 n} + \nu(n))(a_1^* e^{-j\omega_1(n-k)} + a_2^* e^{-j\omega_2(n-k)} + \nu^*(n-k))$$

Taking the expectation gives

$$\begin{aligned} \mathrm{E}[x(n)\; x^*(n-k)] &= \mathrm{E}[\nu(n)\,\nu^*(n-k)] + \mathrm{E}[a_1 a_1^*\, e^{j\omega_1 k}] + \mathrm{E}[a_2 a_2^*\, e^{j\omega_2 k}] \\ &= \sigma_\nu^2\, \delta(k) + \sigma_1^2\, e^{j\omega_1 k} + \sigma_2^2\, e^{j\omega_2 k} \end{aligned}$$

as the expectation of all cross terms will be equal to zero. We can write this in matrix form as

$$\mathbf{R} = \sigma_\nu^2\, \mathbf{I} + \sigma_1^2\, \mathbf{R}_1 + \sigma_2^2\, \mathbf{R}_2 \tag{1}$$

where

$$\mathbf{R}_i = \begin{bmatrix} 1 & e^{j\omega_i} & e^{j2\omega_i} & \cdots & e^{j(N-1)\omega_i} \\ e^{-j\omega_i} & 1 & e^{j\omega_i} & \cdots & e^{j(N-2)\omega_i} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ e^{-j(N-1)\omega_i} & e^{-j(N-2)\omega_i} & e^{-j(N-3)\omega_i} & \cdots & 1 \end{bmatrix}, i = 1, 2.$$

(ii) The eigenvector corresponding to the minimum eigenvalue of $\mathbf{R}$ is obtained by finding $\mathbf{q}_{N-1}$ which minimizes $\mathbf{q}_{N-1}^H \mathbf{R} \mathbf{q}_{N-1}$, subject to the constraint that $\mathbf{q}_{N-1}^H \mathbf{q}_{N-1} = 1$.

$$\mathbf{q}_{N-1}^H \mathbf{R} \mathbf{q}_{N-1} = \sigma_\nu^2 \underbrace{\mathbf{q}_{N-1}^H \mathbf{I} \mathbf{q}_{N-1}}_{=1} + \sigma_1^2 \underbrace{\mathbf{q}_{N-1}^H \mathbf{R}_1 \mathbf{q}_{N-1}}_{\geq 0} + \sigma_2^2 \underbrace{\mathbf{q}_{N-1}^H \mathbf{R}_2 \mathbf{q}_{N-1}}_{\geq 0}$$

The latter inequalities result from the positive-definiteness of $\mathbf{R}_1$ and $\mathbf{R}_2$. Thus $\mathbf{q}_{N-1}^H \mathbf{R} \mathbf{q}_{N-1}$ will be minimized if

$$\mathbf{q}_{N-1}^H \mathbf{R}_1 \mathbf{q}_{N-1} = \mathbf{q}_{N-1}^H \mathbf{R}_2 \mathbf{q}_{N-1} = 0 \tag{2}$$

As was done in Problem P4.7, we can show that any vector orthogonal to the two eigenfilters

$$\mathbf{u}_0 = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & e^{-j\omega_1} & e^{-j2\omega_1} & \cdots & e^{-j(N-1)\omega_1} \end{bmatrix}^T$$

and

$$\mathbf{u}_1 = \frac{1}{\sqrt{N}} \begin{bmatrix} 1 & e^{-j\omega_2} & e^{-j2\omega_2} & \cdots & e^{-j(N-1)\omega_2} \end{bmatrix}^T$$

will satisfy (2). Furthermore, $\mathbf{u}_0$ and $\mathbf{u}_1$ span a 2-dimensional subspace in of the N-dimensional complex vector space. Therefore there are $N-2$ mutually orthonormal vectors which satisfy (2). They are all eigenvectors of $\mathbf{R} = \sigma_\nu^2\, \mathbf{I} + \sigma_1^2\, \mathbf{R}_1 + \sigma_2^2\, \mathbf{R}_2$ each with eigenvalue σ_ν^2.

(iii) Since there is an $(N-2)$-dimensional space spanned by eigenvectors $\mathbf{q}_2$ to $\mathbf{q}_{N-1}$, each with the degenerate eigenvalue $\lambda_2 = \cdots = \lambda_{N-1} = \sigma_\nu^2$, the remaining two eigenvectors must lie within the remaining 2-dimensional subspace which is spanned by $\mathbf{u}_0$ and $\mathbf{u}_1$. We can therefore write these eigenvectors as a linear combination of $\mathbf{u}_0$ and $\mathbf{u}_1$:

$$\mathbf{q}_0 = \alpha_{00}\mathbf{u}_0 + \alpha_{01}\mathbf{u}_1$$

and

$$\mathbf{q}_1 = \alpha_{10}\mathbf{u}_0 + \alpha_{11}\mathbf{u}_1$$

Using the minimax theorem, we may obtain the largest eigenvector $\mathbf{q}_0$, or alternatively the coefficients α_{00} and α_{01}, by maximizing the quantity:

$$\begin{aligned}
\max_{||q_0||=1} \mathbf{q}_0^H \mathbf{R}\,\mathbf{q}_0 &= \max_{||q_0||=1} [\mathbf{q}_0^H(\sigma_\nu^2\mathbf{I})\,\mathbf{q}_0 + \mathbf{q}_0^H(\sigma_1^2\mathbf{R}_1)\,\mathbf{q}_0 + \mathbf{q}_0^H(\sigma_2^2\mathbf{R}_2)\,\mathbf{q}_0] \\
&= \max_{||q_0||=1} [\sigma_\nu^2 + \sigma_1^2(\mathbf{q}_0^H\mathbf{R}_1\,\mathbf{q}_0) + \sigma_2^2(\mathbf{q}_0^H\mathbf{R}_2\,\mathbf{q}_0)] \qquad (3)
\end{aligned}$$

We may similarly obtain the smaller eigenvector in this 2-dimensional subspace $\mathbf{q}_1$, or alternatively the coefficients α_{10} and α_{11} by minimizing the quantity:

$$\begin{aligned}
\min_{||q_1||=1} [\mathbf{q}_1^H \mathbf{R}\,\mathbf{q}_1 &= \min_{||q_1||=1} \mathbf{q}_1^H(\sigma_\nu^2\mathbf{I})\,\mathbf{q}_1 + \mathbf{q}_1^H(\sigma_1^2\mathbf{R}_1)\,\mathbf{q}_1 + \mathbf{q}_1^H(\sigma_2^2\mathbf{R}_2)\,\mathbf{q}_1] \\
&= \min_{||q_1||=1} [\sigma_\nu^2 + \sigma_1^2(\mathbf{q}_1^H\mathbf{R}_1\,\mathbf{q}_1) + \sigma_2^2(\mathbf{q}_1^H\mathbf{R}_2\,\mathbf{q}_1)] \qquad (4)
\end{aligned}$$

Similar to Problem P4.7, noting that

$$\mathbf{R}_1 = \sqrt{N}\begin{bmatrix} \mathbf{u}_0^H \\ e^{-j\omega_1}\mathbf{u}_0^H \\ \vdots \\ e^{-j\omega_1(N-1)}\mathbf{u}_0^H \end{bmatrix} \text{ and } \mathbf{R}_2 = \sqrt{N}\begin{bmatrix} \mathbf{u}_1^H \\ e^{-j\omega_2}\mathbf{u}_1^H \\ \vdots \\ e^{-j\omega_2(N-1)}\mathbf{u}_1^H \end{bmatrix},$$

we have

$$\mathbf{R}_1\mathbf{u}_0 = N\mathbf{u}_0 \quad \text{and} \quad \mathbf{R}_2\mathbf{u}_1 = N\mathbf{u}_1$$

Defining $\beta = \mathbf{u}_0^H\mathbf{u}_1$ we also get

$$\mathbf{R}_1\mathbf{u}_1 = \sqrt{N}\begin{bmatrix} \mathbf{u}_0^H \\ e^{-j\omega_1}\mathbf{u}_0^H \\ \vdots \\ e^{-j\omega_1(N-1)}\mathbf{u}_0^H \end{bmatrix}\mathbf{u}_1 = \sqrt{N}\begin{bmatrix} \beta \\ e^{-j\omega_1}\beta \\ \vdots \\ e^{-j\omega_1(N-1)}\beta \end{bmatrix} = \beta\sqrt{N}\mathbf{u}_0$$

$$\mathbf{R}_2\mathbf{u}_0 = \sqrt{N}\begin{bmatrix} \mathbf{u}_1^H \\ e^{-j\omega_2}\mathbf{u}_1^H \\ \vdots \\ e^{-j\omega_2(N-1)}\mathbf{u}_1^H \end{bmatrix}\mathbf{u}_0 = \sqrt{N}\begin{bmatrix} \beta^* \\ e^{-j\omega_2}\beta^* \\ \vdots \\ e^{-j\omega_2(N-1)}\beta^* \end{bmatrix} = \beta^*\sqrt{N}\mathbf{u}_1$$

Using these equations, we can go about evaluating an expression for the relevant terms in (3).

$$\mathbf{R}_1\mathbf{q}_0 = \mathbf{R}_1(\alpha_{00}\mathbf{u}_0 + \alpha_{01}\mathbf{u}_1) = (\alpha_{00}N + \alpha_{01}\,\beta\sqrt{N})\mathbf{u}_0$$
$$\mathbf{R}_2\mathbf{q}_0 = \mathbf{R}_2(\alpha_{00}\mathbf{u}_0 + \alpha_{01}\mathbf{u}_1) = (\alpha_{00}\,\beta^*\sqrt{N} + \alpha_{01}N)\mathbf{u}_1$$

giving

$$\begin{aligned}
\mathbf{q}_0^H\mathbf{R}_1\mathbf{q}_0 &= (\alpha_{00}N + \alpha_{01}\,\beta\sqrt{N})(\alpha_{00}\mathbf{u}_0 + \alpha_{01}\mathbf{u}_1)^H\mathbf{u}_0 \\
&= (\alpha_{00}N + \alpha_{01}\,\beta\sqrt{N})(\alpha_{00}^* + \alpha_{01}^*\,\beta^*) \\
\mathbf{q}_0^H\mathbf{R}_2\mathbf{q}_0 &= (\alpha_{00}\,\beta^*\sqrt{N} + \alpha_{01}N)(\alpha_{00}\mathbf{u}_0 + \alpha_{01}\mathbf{u}_1)^H\mathbf{u}_1 \\
&= (\alpha_{00}\,\beta^*\sqrt{N} + \alpha_{01}N)(\alpha_{00}^*\,\beta + \alpha_{01}^*)
\end{aligned}$$

by a similar procedure, we can obtain equivalent expressions for the terms in (4).

$$\begin{aligned}
\mathbf{q}_1^H \mathbf{R}_1 \mathbf{q}_1 &= (\alpha_{10} N + \alpha_{11}\, \beta \sqrt{N})(\alpha_{10}^* + \alpha_{11}^*\, \beta^*) \\
\mathbf{q}_1^H \mathbf{R}_2 \mathbf{q}_1 &= (\alpha_{10}\, \beta^* \sqrt{N} + \alpha_{11} N)(\alpha_{10}^*\, \beta + \alpha_{11}^*)
\end{aligned}$$

Maximizing (3) and minimizing (4) with respect to the coefficients $\alpha_{00}, \alpha_{01}, \alpha_{10}, \alpha_{11}$ will result in the two largest eigenvectors.

(iv) When $\mathbf{u}_0^H \mathbf{u}_1 = \beta = 0$, then

$$\begin{aligned}
\mathbf{q}_0^H \mathbf{R}_1 \mathbf{q}_0 = N|\alpha_{00}|^2 \text{ and } \mathbf{q}_0^H \mathbf{R}_2 \mathbf{q}_0 = N|\alpha_{01}|^2 \\
\mathbf{q}_1^H \mathbf{R}_1 \mathbf{q}_1 = N|\alpha_{10}|^2 \text{ and } \mathbf{q}_1^H \mathbf{R}_2 \mathbf{q}_1 = N|\alpha_{11}|^2
\end{aligned}$$

and our maximization equation (3) becomes:

$$\max_{||q_0||=1} \mathbf{q}_0^H \mathbf{R}\, \mathbf{q}_0 = \max_{||q_0||=1} [\sigma_\nu^2 \; + \; N(\sigma_1^2 |\alpha_{00}|^2 \; + \; \sigma_2^2 |\alpha_{01}|^2)]$$

If $\sigma_1^2 > \sigma_2^2$ then $|\alpha_{00}| = 1$ and $|\alpha_{01}| = 0$ maximizes our expression, whereas if $\sigma_1^2 < \sigma_2^2$ then $|\alpha_{00}| = 0$ and $|\alpha_{01}| = 1$ is our solution. In the case when $\sigma_1^2 = \sigma_2^2$ then the expression becomes independent of α_{00} and α_{01} and any unit length vector in our 2-dimensional subspace will be an eigenvector. Similarly for (4).

P4.14. (i)

$$\begin{aligned}
\mathbf{w}^H \mathbf{R} \mathbf{w} &= (\mathbf{w}_R^T - j\mathbf{w}_I^T)(\mathbf{R}_R + j\mathbf{R}_I)(\mathbf{w}_R + j\mathbf{w}_I) \\
&= \mathbf{w}_R^T \mathbf{R}_R \mathbf{w}_R + j\mathbf{w}_R^T \mathbf{R}_R \mathbf{w}_I + j\mathbf{w}_R^T \mathbf{R}_I \mathbf{w}_R - \mathbf{w}_R^T \mathbf{R}_I \mathbf{w}_I \\
&\quad - j\mathbf{w}_I^T \mathbf{R}_R \mathbf{w}_R + \mathbf{w}_I^T \mathbf{R}_R \mathbf{w}_I + \mathbf{w}_I^T \mathbf{R}_I \mathbf{w}_R + j\mathbf{w}_I^T \mathbf{R}_I \mathbf{w}_I
\end{aligned}$$

using the hint that $\mathbf{v}^T \mathbf{R}_I \mathbf{v} = 0$ and noticing the 2nd and 5th terms are negative of each other gives

$$\begin{aligned}
\mathbf{w}^H \mathbf{R} \mathbf{w} &= \mathbf{w}_R^T \mathbf{R}_R \mathbf{w}_R + \mathbf{w}_I^T \mathbf{R}_R \mathbf{w}_I + \mathbf{w}_I^T \mathbf{R}_I \mathbf{w}_R - \mathbf{w}_R^T \mathbf{R}_I \mathbf{w}_I \\
&= \begin{bmatrix} \mathbf{w}_R^T & \mathbf{w}_I^T \end{bmatrix} \begin{bmatrix} \mathbf{R}_R \mathbf{w}_R & -\mathbf{R}_I \mathbf{w}_I \\ \mathbf{R}_I \mathbf{w}_R & \mathbf{R}_R \mathbf{w}_I \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{w}_R^T & \mathbf{w}_I^T \end{bmatrix} \begin{bmatrix} \mathbf{R}_R & -\mathbf{R}_I \\ \mathbf{R}_I & \mathbf{R}_R \end{bmatrix} \begin{bmatrix} \mathbf{w}_R \\ \mathbf{w}_I \end{bmatrix}
\end{aligned}$$

(ii)

$$\mathbf{R}\mathbf{q}_i = (\mathbf{R}_R + j\mathbf{R}_I)(\mathbf{q}_{i,R} + j\mathbf{q}_{i,I}) = \lambda_i \mathbf{q}_i$$

$$\mathbf{R}_R \mathbf{q}_{i,R} + j\mathbf{R}_R \mathbf{q}_{i,I} + j\mathbf{R}_I \mathbf{q}_{i,R} - \mathbf{R}_I \mathbf{q}_{i,I} = \lambda_i (\mathbf{q}_{i,R} + j\mathbf{q}_{i,I})$$

Equating real and imaginary parts of both sides gives

$$\begin{aligned}
\mathbf{R}_R \mathbf{q}_{i,R} - \mathbf{R}_I \mathbf{q}_{i,I} &= \lambda_i \mathbf{q}_{i,R} \\
\mathbf{R}_R \mathbf{q}_{i,I} + \mathbf{R}_I \mathbf{q}_{i,R} &= \lambda_i \mathbf{q}_{i,I}
\end{aligned}$$

which is equivalent to

$$\begin{bmatrix} \mathbf{R}_R & -\mathbf{R}_I \\ \mathbf{R}_I & \mathbf{R}_R \end{bmatrix} \begin{bmatrix} \mathbf{q}_{i,R} \\ \mathbf{q}_{i,I} \end{bmatrix} = \lambda_i \begin{bmatrix} \mathbf{q}_{i,R} \\ \mathbf{q}_{i,I} \end{bmatrix}$$

Also,

$$\mathbf{R}(j\mathbf{q}_i) = \lambda_i (j\mathbf{q}_i)$$