2-1 We use De Morgan's law:
(a) \( \overline{A + B} = \overline{A} + \overline{B} = A + B = A(B + B) = A \)
(b) \( (A + B)(\overline{A + B}) = (A + B)(\overline{A} + \overline{B}) = A\overline{B} + B\overline{A} \)
because \( A\overline{A} = \{\emptyset\} \) \( B\overline{B} = \{\emptyset\} \)

2-2 If \( A = \{2 < x \leq 5\} \) \( B = \{3 < x \leq 6\} \) \( S = \{-\infty < x < \infty\} \) then
\( A + B = \{2 < x \leq 6\} \) \( AB = \{3 < x \leq 5\} \)
\( (A + B)(\overline{A + B}) = \{2 < x \leq 6\} [(x < 3) + \{x > 5\}] \)
\( = \{2 < x < 3\} + \{5 < x \leq 6\} \)

2-3 If \( AB = \{\emptyset\} \) then \( A \subseteq \overline{B} \) hence
\( P(A) \leq P(\overline{B}) \)

2-4 (a) \( P(A) = P(AB) + P(A\overline{B}) \) \( P(B) = P(AB) + P(\overline{AB}) \)
If, therefore, \( P(A) = P(B) = P(AB) \) then
\( P(\overline{AB}) = 0 \) \( P(\overline{AB}) = 0 \) hence
\( P(\overline{A} + \overline{B}) = P(\overline{AB}) + P(\overline{AB}) = 0 \)
(b) If \( P(A) = P(B) = 1 \) then \( 1 = P(A) \leq P(A + B) \) hence
\( 1 = P(A + B) = P(A) + P(B) - P(AB) = 2 - P(AB) \)
This yields \( P(AB) = 1 \)

2-5 From (2-13) it follows that
\( P(A + B + C) = P(A) + P(B + C) - P[A(B + C)] \)
\( P(B + C) = P(B) + P(C) - P(BC) \)
\( P[A(B + C)] = P(AB) + P(AC) - P(ABC) \)
because \( ABAC = ABC \). Combining, we obtain the desired result.
Using induction, we can show similarly that
\( P(A_1 + A_2 + \cdots + A_n) = P(A_1) + P(A_2) + \cdots + P(A_n) \)
\(- P(A_1A_2) - \cdots - P(A_{n-1}A_n) \)
\(+ P(A_1A_2A_3) + \cdots + P(A_{n-2}A_n) \)
\( \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \)
\( \pm P(A_1A_2 \cdots A_n) \)
Any subset of S contains a countable number of elements, hence, it can be written as a countable union of elementary events. It is therefore an event.

Forming all unions, intersections, and complements of the sets {1} and {2,3}, we obtain the following sets:

\{\emptyset, \{1\}, \{4\}, \{2,3\}, \{1,4\}, \{1,2,3\}, \{2,3,4\}, \{1,2,3,4\}\}

If \(A \subseteq B\), \(P(A) = 1/4\), and \(P(B) = 1/3\), then

\[
P(A|B) = \frac{P(AB)}{P(B)} = \frac{P(A)}{P(B)} = \frac{1/4}{1/3} = \frac{3}{4}
\]

\[
P(B|A) = \frac{P(AB)}{P(A)} = \frac{P(A)}{P(A)} = 1
\]

\[
P(A|BC)P(B|C) = \frac{P(ABC)}{P(C)} = \frac{P(AB|C)}{P(C)}
\]

\[
P(A|BC)P(B|C)P(C) = \frac{P(ABC)}{P(C)} \frac{P(BC)}{P(C)} \frac{P(C)}{P(C)} = P(ABC)
\]

We use induction. The formula is true for \(n=2\) because \(P(A_1A_2) = P(A_2|A_1)P(A_1)\). Suppose that it is true for \(n\). Since

\[
P(A_{n+1}A_n \cdots A_1) = P(A_{n+1}|A_n \cdots A_2A_1)P(A_1 \cdots A_n)
\]

we conclude that it must be true for \(n+1\).

The total number of \(m\) element subsets equals \(\binom{n}{m}\) (see Probl. 2-26). The total number of \(m\) element subsets containing \(z_o\) equals \(\binom{n-1}{m-1}\). Hence

\[
p = \binom{n}{m} / \binom{n-1}{m-1} = \frac{m}{n}
\]

Second solution. Clearly, \(P(z_o|A_m) = m/n\) is the probability that \(z_o\) is in a specific \(A_m\). Hence (total probability)

\[
p = \sum P(z_o|A_m)p(A_m) = \frac{m}{n} \sum P(A_m) = \frac{m}{n}
\]

where the summation is over all sets \(A_m\).
2-12 (a) \( P(6 \leq t \leq 8) = \frac{2}{10} \)

(b) \( P(6 \leq t \leq 8|t > 5) = \frac{P(6 \leq t \leq 8)}{P(t > 5)} = \frac{2}{5} \)

2-13 From (2-27) it follows that
\[
P(t_0 \leq t \leq t_0 + t_1|t > t_0) = \int_{t_0}^{t_0 + t_1} \alpha(t) dt / \int_{t_0}^{\infty} \alpha(t) dt
\]
\[
P(t \leq t_1) = \int_{0}^{t_1} \alpha(t) dt
\]
Equating the two sides and setting \( t_1 = t_0 + \Delta t \) we obtain
\[
\frac{\alpha(t_0)}{\int_{t_0}^{\infty} \alpha(t) dt} = \alpha(0)
\]
for every \( t_0 \). Hence,
\[
- \ln \int_{t_0}^{\infty} \alpha(t) dt = \alpha(0)t_0 \quad \int_{t_0}^{\infty} \alpha(t) dt = e^{-\alpha(0)t_0}
\]
Differentiating the setting \( c = \alpha(0) \), we conclude that
\[
\alpha(t_0) = c e^{ct} \quad P(t \leq t_1) = 1 - e^{-ct_1}
\]

2-14 If \( A \) and \( B \) are independent, then \( P(AB) = P(A)P(B) \). If they are mutually exclusive, then \( P(AB) = 0 \). Hence, \( A \) and \( B \) are mutually exclusive and independent iff \( P(A)P(B) = 0 \).
Clearly, \( A_1 = A_1 A_2 + A_1 \bar{A}_2 \) hence

\[
P(A_1) = P(A_1 A_2) + P(A_1 \bar{A}_2)
\]

If the events \( A_1 \) and \( \bar{A}_2 \) are independent, then

\[
P(A_1 \bar{A}_2) = P(A_1) - P(A_1 A_2) = P(A_1) - P(A_1)P(A_2)
\]

\[
= P(A_1)[1 - P(A_2)] = P(A_1)P(\bar{A}_2)
\]

hence, the events \( A_1 \) and \( \bar{A}_2 \) are independent. Furthermore, \( S \) is independent with any \( A \) because \( SA = A \). This yields

\[
P(SA) = P(A) = P(S)P(A)
\]

Hence, the theorem is true for \( n = 2 \). To prove it in general we use induction: Suppose that \( A_{n+1} \) is independent of \( A_1, \ldots, A_n \). Clearly, \( A_{n+1} \) and \( \bar{A}_{n+1} \) are independent of \( B_1, \ldots, B_n \). Therefore

\[
P(B_1 \cdots B_n A_{n+1}) = P(B_1 \cdots B_n)P(A_{n+1})
\]

\[
P(B_1 \cdots B_n \bar{A}_{n+1}) = P(B_1 \cdots B_n)P(\bar{A}_{n+1})
\]

2.16 The desired probabilities are given by (a)

\[
\begin{pmatrix} \frac{m - 1}{k - 1} \\ \frac{n}{k} \end{pmatrix}
\]

(b)

\[
\begin{pmatrix} \frac{m}{k} \\ \frac{n}{k} \end{pmatrix}
\]
2.17 Let \( A_1, A_2 \) and \( A_3 \) represent the events

\[
A_1 = \text{"ball numbered less than or equal to } m \text{ is drawn"}
\]

\[
A_2 = \text{"ball numbered } m \text{ is drawn"}
\]

\[
A_3 = \text{"ball numbered greater than } m \text{ is drawn"}
\]

\[
P(A_1 \text{ occurs } n_1 = k - 1, \ A_2 \text{ occurs } n_2 = 1 \text{ and } A_3 \text{ occurs } n_3 = 0)
= \frac{(n_1 + n_2 + n_3)!}{n_1! n_2! n_3!} \cdot p_1^{n_1} p_2^{n_2} p_3^{n_3}
= \frac{k!}{(k-1)!} \left( \frac{m}{n} \right)^{k-1} \left( \frac{1}{n} \right)
= \frac{k \left( \frac{m}{n} \right)^{k-1}}{n}
\]

2.18 All cars are equally likely so that the first car is selected with probability \( p = 1/3 \). This gives the desired probability to be

\[
\left( \begin{array}{c} 10 \\ 3 \end{array} \right) \left( \frac{1}{3} \right)^3 \left( \frac{2}{3} \right)^7 = 0.26
\]

2.19 \( P\{\text{"drawing a white ball \} } = \frac{m}{m+n} \)

\( P\{\text{atleast one white ball in } k \text{ trials}\})
= 1 - P\{\text{all black balls in } k \text{ trials}\}
= 1 - \frac{\binom{n}{k}}{\binom{m+n}{k}}

2.20 Let \( D = 2r \) represent the penny diameter. So long as the center of the penny is at a distance of \( r \) away from any side of the square, the penny will be entirely inside the square. This gives the desired probability to be

\[
\frac{(1 - 2r)^2}{1} = \left( 1 - \frac{3}{4} \right)^2 = \frac{1}{16}.
\]
2.21 Refer to Example 3.14.
(a) Using (3.39), we get

\[ P(\text{"all one-digit numbers"}) = \frac{\binom{9}{6} \binom{42}{0}}{\binom{51}{6}} = 5 \times 10^{-6}. \]

(b)

\[ P(\text{"two one-digit and four two-digit numbers"}) = \frac{\binom{9}{2} \binom{42}{4}}{\binom{51}{6}} = 0.224. \]

---

2-22 The number of equations of the form \( P(A_i A_k) = P(A_i)P(A_k) \) equals \( \binom{n}{2} \). The number of equations involving \( r \) sets equals \( \binom{n}{r} \). Hence the total number \( N \) of such equations equals

\[ N = \binom{n}{2} + \binom{n}{3} + \ldots + \binom{n}{n} \]

And since

\[ \binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n} = (1+1)^n = 2^n \]

we conclude that

\[ N = 2^n - \binom{n}{0} - \binom{n}{1} = 2^n - 1 - n \]

---

2-23 We denote by \( B_1 \) and \( B_2 \) respectively the balls in boxes 1 and 2 and by \( R \) the set of red balls. We have (assumption)

\[ P(B_1) = P(B_2) = 0.5 \quad P(R|B_1) = 0.999 \quad P(R|B_2) = 0.001 \]

Hence (Bayes' theorem)

\[ P(B_1|R) = \frac{P(R|B_1)P(B_1)}{P(R|B_1)P(B_1) + P(R|B_2)P(B_2)} = \frac{0.999}{0.999 + 0.001} = 0.999 \]
We denote by $B_1$ and $B_2$ respectively the ball in boxes 1 and 2 and by $D$ all pairs of defective parts. We have (assumption)

$$P(B_1) = P(B_2) = 0.5$$

To find $P(D|B_1)$ we proceed as in Example 2-10:

**First solution.** In box $B_1$ there are $1000 \times 999$ pairs. The number of pairs with both elements defective equals $100 \times 99$. Hence,

$$P(D|B_1) = \frac{100 \times 99}{1000 \times 999}$$

**Second solution.** The probability that the first bulb selected from $B_1$ is defective equals $100/1000$. The probability that the second is defective assuming the first was effective equals $99/999$. Hence,

$$P(D|B_1) = \frac{100}{1000} \times \frac{99}{999}$$

We similarly find

$$P(D|B_2) = \frac{100}{2000} \times \frac{99}{1999}$$

(a) $P(D) = P(D|B_1)P(B_1) + P(D|B_2)P(B_2) = 0.0062$

(b) $P(B_1|D) = \frac{P(D|B_1)P(B_1)}{P(D)} = 0.80$

**2-25** Reasoning as in Example 2-13, we conclude that the probability that the bus and the train meet equals

$$(10 + x)60 - \frac{10^2}{2} - \frac{x^2}{2}$$

Equating with 0.5, we find $x = 60 - 10\sqrt{11}$.

**2-26** We wish to show that the number $N_n(k)$ of the element subsets of $S$ equals

$$N_n(k) = \frac{n(n-1) \cdots (n-k+1)}{1 \cdot 2 \cdots k}$$

This is true for $k=1$ because the number of 1-element subsets equals $n$. Using induction in $k$, we shall show that

$$N_n(k+1) = N_n(k) \frac{n-k}{k+1}, \quad 1 < k < n \quad (i)$$

We attach to each $k$-element subset of $S$ one of the remaining $n-k$ elements of $S$. We, then, form $N_n(k)(n-k)$ $k+1$-element subsets. However, these subsets are not all different. They form groups each of which has $k+1$ identical elements. We must, therefore, divide by $k+1$. 


In this experiment we have 8 outcomes. Each outcome is a selection of a particular coin and a specific sequence of heads or tails; for example fhh is the outcome "we selected the fair coin and we observed hh". The event $F = \{\text{the selected coin is fair}\}$ consists of the four outcomes fhh, fht, fth and fhh. Its complement $\overline{F}$ is the selection of the two-headed coin. The event $HH = \{\text{heads at both tosses}\}$ consists of two outcomes. Clearly,

$$P(F) = P(\overline{F}) = \frac{1}{2}, \quad P(HH|F) = \frac{1}{4}, \quad P(HH|\overline{F}) = 1$$

Our problem is to find $P(F|HH)$. From (2-41) and (2-43) it follows that

$$P(HH) = P(HH|F)P(F) + P(HH|\overline{F})P(\overline{F}) = \frac{5}{8}$$

$$P(F|HH) = \frac{P(HH|F)P(F)}{P(HH)} = \frac{\frac{1}{4} \times \frac{1}{2}}{\frac{5}{8}} = \frac{1}{5}$$
3.1 (a) \( P(\text{A occurs at least twice in } n \text{ trials}) \)
\[ = 1 - P(\text{A never occurs in } n \text{ trials}) - P(\text{A occurs once in } n \text{ trials}) \]
\[ = 1 - (1 - p)^n - np(1 - p)^{n-1} \]

(b) \( P(\text{A occurs at least thrice in } n \text{ trials}) \)
\[ = 1 - P(\text{A never occurs in } n \text{ trials}) - P(\text{A occurs once in } n \text{ trials}) \]
\[ - P(\text{A occurs twice in } n \text{ trials}) \]
\[ = 1 - (1 - p)^n - np(1 - p)^{n-1} - \frac{n(n-1)}{2} p^2(1 - p)^{n-2} \]

3.2
\[ P(\text{double six}) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36} \]

\( P(\text{"double six at least three times in } n \text{ trials"}) \)
\[ = 1 - \binom{50}{0} \left( \frac{1}{36} \right)^0 \left( \frac{35}{36} \right)^{50} - \binom{50}{1} \left( \frac{1}{36} \right)^1 \left( \frac{35}{36} \right)^{49} - \binom{50}{2} \left( \frac{1}{36} \right)^2 \left( \frac{35}{36} \right)^{48} \]
\[ = 0.162 \]

3-3 If \( A = \{\text{seven}\} \), then
\[ P(A) = \frac{6}{36}, \quad P(\bar{A}) = \frac{5}{6} \]

If the dice are tossed 10 times, then the probability that \( \bar{A} \) will occur 10 times equals \( (5/6)^{10} \). Hence, the probability \( p \) that \( \{\text{seven}\} \) will show at least once equals
\[ 1 - (5/6)^{10} \]
3-4 If $k$ is the number of heads, then

$$P\{\text{even}\} = P\{k = 0\} + P\{k = 2\} + \cdots$$

$$= q^n + \binom{n}{2} p^2 q^{n-2} + \binom{n}{4} p^4 q^{n-4} + \cdots$$

But

$$1 = (q + q)^n = q^n + \binom{n}{1} p q^{n-1} + \binom{n}{2} p^2 q^{n-2} + \cdots$$

$$= \binom{n}{1} p q^{n-1} + \binom{n}{2} p^2 q^{n-2} - \cdots$$

Adding, we obtain

$$1 + (p - q)^n = 2 P\{\text{even}\}$$

3-5 In this experiment, the total number of outcomes is the number $\binom{N}{n}$ of ways of picking $n$ out of $N$ objects. The number of ways of picking $k$ out of the $K$ good components equals $\binom{K}{k}$ and the number of ways of picking $n-k$ out of the $N-K$ defective components equals $\binom{N-K}{n-k}$. Hence, the number of ways of picking $k$ good components and $n-k$ defective components equals $\binom{K}{k} \binom{N-K}{n-k}$. From this and (2-25) it follows that

$$p = \binom{K}{k} \binom{N-K}{n-k} / \binom{N}{n}$$

3.6 (a)

$$p_1 = 1 - \left( \frac{5}{6} \right)^6 = 0.665$$

(b)

$$1 - \left( \frac{5}{6} \right)^{12} - \left( \frac{12}{1} \right) \left( \frac{1}{6} \right) \left( \frac{5}{6} \right)^{11} = 0.619$$

(c)

$$1 - \left( \frac{5}{6} \right)^{18} - \left( \frac{18}{1} \right) \left( \frac{1}{6} \right) \left( \frac{5}{6} \right)^{17} - \left( \frac{18}{2} \right) \left( \frac{1}{6} \right)^2 \left( \frac{5}{6} \right)^{16} = 0.597$$
3.7 (a) Let $n$ represent the number of wins required in 50 games so that the net gain or loss does not exceed $1$. This gives the net gain to be

$$-1 < n - \frac{50}{4} < 1$$
$$16 < n < 17.3$$
$$n = 17$$

$$P(\text{net gain does not exceed } $1) = \left(\frac{50}{17}\right)^{17} \left(\frac{3}{4}\right)^{33} = 0.432$$

$$P(\text{net gain or loss exceeds } $1) = 1 - 0.432 = 0.568$$

(b) Let $n$ represent the number of wins required so that the net gain or loss does not exceed $5$. This gives

$$-5 < n - \frac{(50 - n)}{2} < 5$$
$$13.3 < n < 20$$

$$P(\text{net gain does not exceed } $5) = \sum_{n=14}^{19} \binom{50}{n} \left(\frac{1}{4}\right)^n \left(\frac{3}{4}\right)^{50-n} = 0.349$$

$$P(\text{net gain or loss exceeds } $5) = 1 - 0.349 = 0.651$$
3.8 Define the events

\[ A = \text{"r successes in } n \text{ Bernoulli trials"} \]

\[ B = \text{"success at the } i^{th} \text{ Bernoulli trial"} \]

\[ C = \text{"r} - 1 \text{ successes in the remaining } n - 1 \text{ Bernoulli trials excluding the } i^{th} \text{ trial"} \]

\[ P(A) = \binom{n}{r} p^r q^{n-r} \]

\[ P(B) = p \]

\[ P(C) = \binom{n-1}{r-1} p^{r-1} q^{n-r} \]

We need

\[ P(B|A) = \frac{P(AB)}{P(A)} = \frac{P(BC)}{P(A)} = \frac{P(B)P(C)}{P(A)} = \frac{r}{n}. \]

3.9 There are \( \binom{52}{13} \) ways of selecting 13 cards out of 52 cards. The number of ways to select 13 cards of any suit (out of 13 cards) equals \( \binom{13}{13} = 1 \). Four such (mutually exclusive) suits give the total number of favorable outcomes to be 4. Thus the desired probability is given by

\[ \frac{4}{\binom{52}{13}} = 6.3 \times 10^{-12} \]
3.10 Using the hint, we obtain

\[ p(N_{k+1} - N_k) = q(N_k - N_{k-1}) - 1 \]

Let

\[ M_{k+1} = N_{k+1} - N_k \]

so that the above iteration gives

\[
M_{k+1} = \frac{q}{p} M_k - \frac{1}{p} \left( \begin{array}{l}
(\frac{q}{p}) M_1 - \frac{1}{p-q} \left\{ 1 - (\frac{q}{p})^i \right\}, \quad p \neq q \\
M_1 - \frac{k}{p}, \quad p = q
\end{array} \right)
\]

This gives

\[
N_i = \sum_{k=0}^{i-1} M_{k+1}
\]

\[
= \left\{ \begin{array}{l}
(M_1 + \frac{1}{p-q}) \sum_{k=0}^{i-1} \left( \frac{q}{p} \right)^k - \frac{i}{p-q}, \quad p \neq q \\
iM_1 - \frac{i(i-1)}{2p}, \quad p = q
\end{array} \right.
\]

where we have used \( N_a = 0 \). Similarly \( N_{a+b} = 0 \) gives

\[
M_1 + \frac{1}{p-q} = \frac{a+b}{p-q} \cdot \frac{1 - q/p}{1 - (q/p)^{a+b}}.
\]

Thus

\[
N_i = \left\{ \begin{array}{l}
\frac{a+b}{p-q} \cdot \frac{1 - (q/p)^i}{1 - (q/p)^{a+b}} - \frac{i}{p-q}, \quad p \neq q \\
i(a+b-i), \quad p = q
\end{array} \right.
\]
which gives for $i = a$

$$N_a = \begin{cases} \frac{a + b}{p - q} \cdot \frac{1 - (q/p)^a}{1 - (q/p)^{a+b}} - \frac{a}{p - q}, & p \neq q \\ ab, & p = q \end{cases}$$

$$= \begin{cases} \frac{b}{2p - 1} - \frac{a + b}{2p - 1} \cdot \frac{1 - (q/p)^b}{1 - (q/p)^{a+b}} - \frac{a}{p - q}, & p \neq q \\ ab, & p = q \end{cases}$$

3.11

$$P_n = pP_{n+\alpha} + qP_{n-\beta}$$

Arguing as in (3.43), we get the corresponding iteration equation

$$P_n = P_{n+\alpha} + qP_{n-\beta}$$

and proceed as in Example 3.15.

3.12 Suppose one bet on $k = 1, 2, \cdots, 6$.

Then

$$p_1 = P(k \text{ appears on one dice}) = \binom{3}{1} \left(\frac{1}{6}\right) \left(\frac{5}{6}\right)^2$$

$$p_2 = P(k \text{ appear on two dice}) = \binom{3}{2} \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)$$

$$p_3 = P(k \text{ appear on all the tree dice}) = \left(\frac{1}{6}\right)^3$$

$$p_0 = P(k \text{ appear none}) = \left(\frac{5}{6}\right)^3$$

Thus, we get

$$\text{Net gain} = 2p_1 + 3p_2 + 4p_3 - p_0 = 0.343.$$
CHAPTER 4

4-1 From the evenness of \( f(x) \): 1 - \( F(x) = F(-x) \).

From the definition of \( x_u \): \( u = F(x_u) \), 1 - \( u = F(x_{1-u}) \). Hence

\[
1 - u = 1 - F(x_u) = F(-x_u) = F(x_{1-u}) \quad \Rightarrow \quad x_u = x_{1-u}
\]

4-2 From the symmetry of \( f(x) \): 1 - \( F(\eta + a) = F(\eta - a) \). Hence [see (4-8)]

\[
P(\eta - a < x < \eta + a) = F(\eta + a) - F(\eta - a) = 2F(\eta + a) - 1
\]

This yields

\[
1 - \alpha = 2F(\eta + a) - 1 \quad \Rightarrow \quad F(\eta + a) = 1 - \alpha/2 \quad \eta + a = x_{1-\alpha/2}
\]

\[
F(\eta - \eta) = \alpha/2 \quad a - \eta = x_{\alpha/2}
\]

4-3 (a) In a linear interpolation:

\[
x_u \approx x_a + \frac{x_b - x_a}{u_b - u_a} (u - u_a) \quad \text{for} \quad x_a < x_u < x_b
\]

From Table 4-1 page 106

\[
z_{0.9} \approx 1.25 + \frac{0.00565}{0.00885} \times 0.05 = 1.2819
\]

Proceeding similarly, we obtain

<table>
<thead>
<tr>
<th>( u )</th>
<th>0.9</th>
<th>0.925</th>
<th>0.95</th>
<th>0.975</th>
<th>0.99</th>
</tr>
</thead>
<tbody>
<tr>
<td>( z_u )</td>
<td>1.282</td>
<td>1.440</td>
<td>1.645</td>
<td>1.960</td>
<td>2.327</td>
</tr>
</tbody>
</table>

(b) If \( z \) is such that \( x = \eta + \sigma z \) then \( z \) is \( N(0,1) \) and \( G(z) = F_x(\eta + \sigma z) \). Hence,

\[
u = G(z_u) = F_x(\eta + \sigma z_u) = F_x(x_u) \quad \Rightarrow \quad x_u = \eta + \sigma z_u
\]
4-4 \( p_k - 2G(k) = 1 = 2 \text{erf}k \)

(a) From Table 4-1

<table>
<thead>
<tr>
<th>( k )</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p_k )</td>
<td>0.6827</td>
<td>0.9545</td>
<td>0.9973</td>
</tr>
</tbody>
</table>

(b) From Table 3-1 with linear interpolation:

| \( p_k \) | 0.9   | 0.99 | 0.999 |
| \( k \) | 1.282 | 2.32 | 3.090 |

(c) \( P(\eta - z_u \sigma < x < \eta + z_u \sigma) = 2G(z_u) - 1 = \gamma \)

Hence, \( G(z_u) = (1+\gamma)/2 \) \( u = (1+\gamma)/2 \)

4-5 (a) \( F(x) = x \) for \( 0 \leq x \leq 1 \); hence, \( u = F(x_u) = x_u \)

(b) \( F(x) = 1 - e^{-2x} \) for \( x \geq 0 \); hence, \( u = 1 - e^{-2x_u} \)

\[ x_u = \frac{1}{2} \ln(1-u) \]

<table>
<thead>
<tr>
<th>( u )</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_u )</td>
<td>0.0527</td>
<td>0.1116</td>
<td>0.1783</td>
<td>0.2554</td>
<td>0.3466</td>
<td>0.4581</td>
<td>0.6020</td>
<td>0.847</td>
<td>1.1513</td>
</tr>
</tbody>
</table>

4-6 Percentage of units between 96 and 104 ohms equals 100p where \( p = P(96 < R < 104) = F(104) - F(96) \)

(a) \( F(R) = 0.1(R-95) \) for \( 95 \leq R \leq 105 \). Hence,

\[ p = 0.1(104-95) - 0.1(96-95) = 0.8 \]

(b) \( p = G(2.5) - G(-2.5) = 0.9876 \)

4-7 From (4-34), with \( \alpha = 2 \) and \( \beta = 1/\lambda \) we get \( f(x) = c^2 x e^{-cx}U(x) \)

\[ F(x) = c^2 \int_0^x y e^{-cy}dy = 1 - e^{-cx} - cxe^{-cx} \]
\{(x - 10)^2 < 4\} = \{8 < x < 12\}

\(P\{(x - 10)^2 < 4\} = G(12 - 10) - G(8 - 10) = 0.954\)

\[f(x \mid (x - 10)^2 < 4) = \frac{f(x)}{P(8 < x < 12)} = \frac{1}{0.954 \sqrt{2\pi}} e^{-\frac{(x-10)^2}{2}}\]

for \(8 < x < 12\) and zero otherwise

\[F(x) = (1 - e^{-\alpha x})U(x-c) \quad f(x) = (1 - e^{-\alpha c})\delta(x-c) + e^{-\alpha x}U(x-c)\]

4-10

(a) \(P\{1 \leq x \leq 2\} = G\left(\frac{2}{2}\right) - G\left(\frac{1}{2}\right) = 0.1499\)

(b) \(P\{1 \leq x \leq 2 \mid x \geq 1\} = \frac{G(1) - G(0.5)}{1 - G(0.5)} = \frac{0.1499}{0.3085} = 0.4857\)

because \(\{1 \leq x \leq 2, x \geq 1\} = \{1 \leq x \leq 2\}\)

4-11

If \(x(t_1) \leq x\) then \(t_1 \leq y = G(x)\)

Hence,

\(P\{x \leq x\} = P\{t_1 \leq y\} = y = G(x)\)
4-12 (a) \[ P\{x < 1024\} = G\left(\frac{1024 - 1000}{20}\right) = G(1.2) = 0.8849 \]

(b) \[ P\{x < 1024|x > 961\} = \frac{P\{961 < x < 1024\}}{P\{x > 961\}} = \frac{G(1.2) - G(1.95)}{1 - G(1.95)} = 0.8819 \]

(c) \[ P\{31 < \sqrt{x} < 32\} = P\{961 < x < 1024\} = 0.8593 \]

4-13 \[ P\{x = 0\} = \frac{1}{8} \quad P\{x = 1\} = \frac{3}{8} \quad P\{x = 2\} = \frac{3}{8} \quad P\{x = 3\} = \frac{1}{8} \]

4-14 (a) \[ f_x(x) = \frac{1}{2 \cdot 900} \sum_{k=0}^{900} \binom{900}{k} \delta(x-k) \]

(b) \[ f_x(x) = \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{900} e^{-\frac{(k-450)^2}{450}} \delta(x-k) \]

(b) \[ P\{435 \leq x \leq 460\} = G\left(\frac{10}{15}\right) - G\left(-\frac{15}{15}\right) = 0.5888 \]

4-15 If \( x > b \) then \( \{x \leq x\} = S \quad F(x) = 1 \)
If \( x < a \) then \( \{x \leq x\} = \emptyset \quad F(x) = 0 \)

4-16 If \( y(t_1) \leq w \), then \( x(t_1) \leq w \) because \( x(t_1) \leq y(t_1) \).

Hence,
\[ \{y \leq w\} \subset \{x \leq w\} \quad P\{y \leq w\} \leq P\{x \leq w\} \]

Therefore \[ F_y(w) \leq F_x(w) \]
4-17 From (4-80)
\[
f(x) = kx e^{-kx^2/2}
\]

4-18 It follows from (2-41) with
\[
A_1 = \{x < x\} \quad A_2 = \{x > x\}
\]

4-19 It follows from
\[
F_x(x|A) = \frac{P\{x \leq x, A\}}{P(A)} \quad P(A|x \leq x) = \frac{P\{x \leq x, A\}}{P\{x \leq x\}}
\]

4-20 We replace in (4-80) all probabilities with conditional probabilities assuming \(\{x \leq x_o\}\). This yields
\[
\int_{-\infty}^{\infty} P(A|x = x, x \leq x_o)f(x|x \leq x_o)dx = P(A|x \leq x_o)
\]
But \(f(x|x \leq x_o) = 0\) for \(x > x_o\) and \(\{x = x, x \leq x_o\} = \{x = x\}\) for \(x \leq x_o\). Hence,
\[
\int_{-\infty}^{x_o} P(A|x = x)f(x|x \leq x_o)dx = P(A|x \leq x_o)
\]
Writing a similar equation for \(P(B|x \leq x_o)\) we conclude that, if \(P(A|x = x) = P(B|x = x)\) for \(x \leq x_o\), then \(P(A|x \leq x_o) = P(B|x \leq x_o)\)
4-21 (a) Clearly, \( f(p) = 1 \) for \( 0 \leq p \leq 1 \) and 0 otherwise; hence

\[
P(0.3 \leq p \leq 0.7) = \int_{0.3}^{0.7} dp = 0.4
\]

(b) We wish to find the conditional probability \( P(0.3 \leq p \leq 0.7|A) \) where \( A = \{6 \text{ heads in 10 tosses}\} \). Clearly \( P(A|p=p) = p^6(1-p)^4 \). Hence, [see (4-81)]

\[
f(p|A) = \frac{p^6(1-p)^4}{\int_0^1 p^6(1-p)^4 dp} = \frac{p^6(1-p)^4}{4329 \times 10^{-7}}
\]

This yields

\[
P(0.3 \leq p \leq 0.7|A) = \int_{0.3}^{0.7} f(p|A) dp = \frac{10^7}{4329} \int_{0.3}^{0.7} p^6(1-p)^4 dp = 0.768
\]

4-22 (a) In this problem, \( f(p) = 5 \) for \( 0.4 \leq p \leq 0.6 \) and zero otherwise; hence [see (4-82)]

\[
P(H) = 5 \int_{0.4}^{0.6} p dp = 0.5
\]

(b) With \( A = \{60 \text{ heads in 100 tosses}\} \) it follows from (4-82) that

\[
f(p|A) = \frac{p^{60}(1-p)^{40}}{\int_{0.4}^{0.6} p^{60}(1-p)^{40} dp}
\]

for \( 0.4 \leq p \leq 0.6 \) and 0 otherwise. Replacing \( f(p) \) by \( f(p|A) \) in (4-82), we obtain

\[
P(H|A) = \int_{0.4}^{0.6} pf(p|A) dp = 0.56
\]
For a fair coin \( \sqrt{npq} = \sqrt{n}/2 \). If

\[ k_1 = 0.49n \quad \text{and} \quad k_2 = 0.52n \]

then

\[
\frac{k_2 - np}{\sqrt{npq}} = \frac{0.52n - n/2}{\sqrt{n}/2} = 0.04\sqrt{n} \quad \quad \frac{k_1 - np}{\sqrt{npq}} = -0.02\sqrt{n}
\]

\[ P(k_1 \leq k \leq k_2) = G(0.04\sqrt{n}) + G(0.02\sqrt{n}) - 1 \geq 0.9 \]

From Table 4-1 (page 106) it follows that

\[ 0.02\sqrt{n} > 1.3 \quad \quad n > 65^2 \]
(a) Assume \( n = 1,000 \) (Note correction to the problem)

\[
P(A) = 0.6 \quad np = 600 \quad npq = 240 \quad k_2 = 650 \quad k_1 = 550
\]

\[
\frac{k_2 - np}{\sqrt{npq}} = \frac{50}{\sqrt{240}} = 3.23 \quad \frac{k_1 - np}{\sqrt{npq}} = -3.23
\]

\[
P(550 \leq k \leq 650) = 2G(3.23) - 1 = 0.999
\]

(b) \( P(0.59n \leq k \leq 0.61n) = 2G\left(\frac{0.01n}{\sqrt{0.24n}}\right) - 1 \)

\[
= 2G\left(\frac{n}{\sqrt{2400}}\right) - 1 = 0.476
\]

Hence, (Table 3-1) \( n = 9220 \)

---

(b) With \( a = 0, b = T/4 \) it follows that

\[
p = 1 - e^{-1/4} = 0.22 \quad np = 220 \quad npq = 171.6 \quad k_2 = 100
\]

\[
\frac{k_2 - np}{\sqrt{npq}} = -9.16 \quad \text{and (4-100) yields}
\]

\[
P(0 \leq k \leq 100) = G(-9.16) = 0.
\]

---

The event

\( A = \{ \text{k heads show at the first n tossings but not earlier} \} \)

occurs iff the following two events occur

\( B = \{ \text{k-1 heads show at the first n-1 tossing} \} \)

\( C = \{ \text{heads show at the nth tossing} \} \)

And since these two events are independent and

\[
P(B) = \binom{n-1}{k-1} p^{k-1} q^{n-1-(k-1)} \quad P(C) = p
\]

we conclude that

\[
P(A) = P(B)P(C) = \binom{n-1}{k-1} p^k q^{n-k}
\]
\[ - \frac{d}{dx} \left( \frac{1}{x} e^{-x^2/2} \right) = \left(1 + \frac{1}{x^2} \right) e^{-x^2/2} > e^{-x^2/2} \]

Multiplying by \(1/\sqrt{2\pi}\) and integrating from \(x\) to \(\infty\), we obtain

\[ \frac{1}{\sqrt{2\pi}} e^{-x^2/2} > \frac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-\zeta^2/2} \, d\zeta = 1 - G(x) \]

because

\[ \frac{1}{x} e^{-x^2/2} \to 0 \quad \text{as} \quad x \to \infty \]

The first inequality follows similarly because

\[- \frac{d}{dx} \left[ \left(\frac{1}{x} - \frac{1}{x^3} \right) e^{-x^2/2} \right] = \left(1 - \frac{3}{4} \right) e^{-x^2/2} < e^{-x^2/2} \]

---

4-29 If \(P(A) = p\) then \(P(\overline{A}) = 1-p\). Clearly \(P_1 = 1-Q_1\) where \(Q_1\) equals the probability that \(A\) does not occur at all. If \(pn \ll 1\), then \(Q_1 = (1-p)^n = 1 - np \quad P_1 = p\)

4-30 With \(p = 0.02, n = 100, k = 3\), it follows from (4-107) that the unknown probability equals

\[ \binom{100}{3}(0.02)^3(0.98)^97 = \frac{2^3}{3!} e^{-2} = \frac{4}{3} e^{-2} \]

4-31 With \(n = 3, r = 3, k_1 = 2, k_2 = 2, k_3 = 1, p_1 = p_2 = p_3 = 1/6, \) it follows from (4-102) that the unknown probability equals

\[ \frac{1}{\binom{10}{3} \cdot 3!} \frac{1}{6} = 0.00386 \]

4-32 With \(t = 2, k_1 = k, k_2 = n-k, p_1 = p, p_2 = 1-p = q\), we obtain

\[ k_1 - np_1 = k - np \quad k_2 - np_2 = n-k-nq = np - k \]

Hence, the bracket in (4-103) equals

\[ \frac{(k_1-np_1)^2}{np_1} + \frac{(k_2-np_2)^2}{np_2} = \frac{(k-np)^2}{n} \left( \frac{1}{p} + \frac{1}{q} \right) = \frac{(k-np)^2}{n p q} \]

as in (4-90).
4-33  \[ P(M) = 2/36 \quad P(\overline{M}) = 34/36. \] The events \( M \) and \( \overline{M} \) form a partition, hence, [see (2-41)]

\[
P(A) = P(A|M)P(M) + P(A|\overline{M})P(\overline{M})
\]

Clearly, \( P(A|M) = 1 \) because, if \( M \) occurs at first try, \( X \) wins. The probability that \( X \) wins after the first try equals \( P(A|\overline{M}) \). But in the experiment that starts at the second rolling, the first player is \( Y \) and the probability that he wins equals \( P(\overline{A}) = 1-p \). Hence, \( P(A|\overline{M}) = P(\overline{A}) = 1-p \). And since \( P(M) = 1/18 \)

\[ P(M) = 17/18 \quad P(\overline{M}) = 18/35 \]

4-34

(a) Each of the \( n \) particles can be placed in any one of the \( m \) boxes. There are \( n \) particles, hence, the number of possibilities equals \( N = m^n \). In the \( m \) preselected boxes, the particles can be placed in \( N_A = n \)! ways (all permutations of \( n \) objects). Hence \( p = n!/m^n \).

(b) \[ \begin{array}{cccccc}
\times & & \times & & \times & \times \quad \text{n particles} \\
\times & & \times & & \times & \text{m-1 interior walls} \\
\end{array} \]

All possibilities are obtained by permuting the \( n+m-1 \) objects consisting of the \( m-1 \) interior walls with and \( n \) particles. The \( (m-1)! \) permutations of the walls and the \( n! \) permutations of the particles must count as one. Hence

\[ N = \frac{(m+m-1)!}{m! (m-1)!} \quad N_A = 1 \]

(c) Suppose that \( S \) is a set consisting of the \( m \) boxes. Each placing of the particles specifies a subset of \( S \) consisting of \( n \) elements (box). The number of such subsets equals \( \binom{m}{n} \) (see Prob. 2-26). Hence,

\[ N = \binom{m}{n} \quad N_A = 1 \]
4-35 If $k_1 + k_2 \ll n$, then $k_3 \simeq n$ and
\[
k_3(p_1 + p_2) = \left[ n - (k_1 + k_2) \right] (p_1 + p_2) = n(p_1 + p_2)
\]
\[
p_3 = 1 - (p_1 + p_2) = e^{-(p_1 + p_2)}
\]
\[
p_3 \simeq e^{-(p_1 + p_2)}
\]
\[
\frac{n!}{k_1!k_2!k_3!} = \frac{n(n-1) \cdots (n-k_3+1)}{k_1!k_2!k_3!} = \frac{n!}{k_1!k_2!}
\]

Hence,
\[
\frac{n!}{k_1!k_2!k_3!} \frac{k_1}{p_1} \frac{k_2}{p_2} \frac{k_3}{p_3} = e^{-(np_1)(np_1)} e^{-(np_2)} \frac{k_1}{k_1!} \frac{k_2}{k_2!}
\]

4-36 The probability $p$ that a particular point is in the interval $(0,2)$ equals $2/100$. (a) From (3-13) it follows that the probability $p_1$ that only one out of the 200 points is in the interval $(0,2)$ equals
\[
p_1 = \binom{200}{1} \times 0.02 \times 0.09^{199}
\]

(b) With $np = 200 \times 0.02 = 4$ and $k = 1$, (3-41) yields $p_1 \simeq e^{-4} \times 4 = 0.073$
CHAPTER 5

5-1 \[ \eta = 2\eta_x + 4 = 14 \quad \sigma_y^2 = 4\sigma_x^2 = 16 \]

5-2 \[ y \leq y = (-4x + 3 \leq y) \quad (x \leq (y - 3)/4) \]. Hence

\[ F_y(y) = P \left\{ x \geq \frac{3-y}{4} \right\} = 1 - F_x \left( \frac{3-y}{4} \right) \quad f_y(y) = \frac{1}{4} f_x \left( \frac{3-y}{4} \right) \]

Since \( F_x(x) = (1-e^{-2x})U(x) \), this yields

\[ F_y(y) = e^{(y-3)/2}U \left( \frac{y-3}{2} \right) \quad f_y(y) = \frac{1}{2} e^{(y-3)/2}U \left( \frac{y-3}{2} \right) \]

5-3 From Example 5-3 with \( F_x = G(x/c) \):

\[ f_y(y) = \begin{cases} G(y/c + 1) & y \geq 0 \\ G(y/c - 1) & y < 0 \end{cases} \]

\[ f_y(y) = 0.68 \delta(y) + \frac{1}{c\sqrt{2\pi}} \left[ e^{-(y+c)^2/2c^2}U(y) + e^{-(y-c)^2/2c^2}U(-y) \right] \]

26
5-4 If \( y = x^2 \) and \( F_x(x) = \frac{(x+2c)}{4c} \) for \( |x| \leq 2c \), then (see Example 5-2) \( F_y(y) = \frac{\sqrt{y}}{2c} \) and \( f_y(y) = \frac{1}{4\sqrt{y}} \) for \( 0 < y < 2c \).

![](image1.png)

5-5 From Example 5-4 with \( F_x(x) = G(x/b) \): For \( |x| \leq b \), \( F_y(y) = G(y/b) \) and

\[
f_y(y) = 0.166(y+b) + \frac{1}{b\sqrt{2\pi}} e^{-y^2/2b^2} + 0.166(y-b)
\]

![](image2.png)

5-6 The equation \( y = -\ln x \) has a single solution \( x = e^{-y} \) for \( y > 0 \) and no solutions for \( y < 0 \). Furthermore, \( g'(x) = -1/x = -e^y \). Hence

\[
f_y(y) = \frac{f_x(e^{-y})}{e^y} \quad U(y) = e^y U(y)
\]
Clearly, \( z \leq z \) iff the number \( n(0,z) \) of the points in the interval \( (0,z) \) is at least one.

Hence,

\[
F_z(z) = P(z \leq z) = P(n(0,z) > 0) = 1 - P(n(0,z) = 0)
\]

The probability \( p \) that a particular point is in the integral \( (0,z) \) equals \( z/100 \). With \( n = 200 \), \( k = 0 \), and \( p = z/100 \), (3-21) yields \( P(n(0,z) = 0) = (1-p)^{200} \). Hence,

(a) \[
F_z(z) = 1 - \left(1 - \frac{z}{100}\right)^{100}
\]

(b) From (4-107) it follows that \( F_z(z) \approx 1 - e^{-2z} \) for \( z \ll 100 \).

\[5.8\]

\[
Y = \sqrt{X} \quad \Rightarrow \quad x_1 = y^2
\]

Thus

\[
f_Y(y) = \left| \frac{dy}{dx} \right| f_X(x_1) = 2yf_X(y^2)
\]

\[
\frac{2y}{\lambda} e^{-y^2/\lambda} = \begin{cases} 
\frac{y}{\sigma^2} e^{-y^2/2\sigma^2}, & y > 0 \\
0, & \text{otherwise}
\end{cases}
\]

which represents Rayleigh density function (with \( \lambda = 2\sigma^2 \)).

\[5.9\]

For both cases, \( f_y(y) = 0 \) for \( y < 0 \).

(a) If \( y > 0 \) and \( |x| = y \), then \( x_1 = y, x_2 = -y \). Hence

\[
f_y(y) = [f_x(y) + f_x(-y)]u(y)
\]

(b) If \( y > 0 \) and \( e^{-x}u(x) = y \), then \( x = -ny \).

Furthermore, \( P(y = 0) = P(x < 0) = F_x(0) \). Hence

\[
f_y(y) = F_x(0) \delta(y) + \frac{1}{y} f_x(-ny)u(y)
\]
5-10 (a) If \( y \geq 0 \) and \( (x - 1)U(x - 1) = y \), then \( \{ y \leq y \} = \{ x \leq y + 1 \} \). If \( y < 0 \), then \( \{ y \leq y \} = \{ \emptyset \} \).

\[
F_y(y) = F_x(1 + y)U(y) = [1 - e^{-2(y + 1)}]U(y)
\]

\[
f_y(y) = (1 - e^{-2})\delta(y) + 2e^{-2(y + 1)}U(y)
\]

(b) If \( y > 0 \) and \( y = x^2 \), then \( \{ y \leq y \} = \{ -\sqrt{y} \leq x \leq \sqrt{y} \} \)

\[
F_y(y) = F_x(\sqrt{y}) - F_x(-\sqrt{y}) = (1 - e^{-2\sqrt{y}})U(y)
\]

\[
f_y(y) = \frac{1}{\sqrt{y}}e^{-2\sqrt{y}}U(y)
\]

5-11

If \( y = \arctan x \), then \( \frac{dy}{dx} = \frac{1}{1 + x^2} \)

\[
f_y(y) = (1 + x^2)f_x(\tan y) = \frac{1 + x^2}{\pi(1 + x^2)} = \frac{1}{\pi} \quad \frac{\pi}{2} < y < \frac{\pi}{2}
\]

5-12

(a) If \( y = x^3 \) then \( x = \frac{3}{y} \) for any \( y \)

\[
f_y(y) = \frac{1}{3} f_x(\frac{3}{y}) = \frac{1}{12\pi^3 y^2}
\]

for \( |y| < 8\pi^3 \) and zero otherwise

(b) If \( y = x^4 \) and \( y > 0 \), then \( x_1 = \frac{y}{\sqrt{y}} \quad x_1 = -\frac{y}{\sqrt{y}} \)

\[
f_y(y) = \frac{1}{4y^3} \left[ f_x(y) + f_x(-\frac{y}{\sqrt{y}}) \right] = \frac{1}{8\pi^4 y^3}
\]

for \( 0 < y < 4\pi^2 \) and zero otherwise

(c) If \( y = 2 \sin (3x + 40^\circ) \) and \( |y| < 2 \) then \( x = x_i \) as shown.

\[
\frac{dy}{dx} = \frac{1}{6\sqrt{1 - y^2}}
\]

In the interval \((-2\pi, 2\pi)\) there are 12 \( x_i \)'s. Hence

\[
f_y(y) = \frac{1}{3\sqrt{4 - y^2}} \quad \sum_{i=1}^{12} f_x(x_i) = \frac{12}{12\pi\sqrt{4 - y^2}} = \frac{1}{\pi\sqrt{4 - y^2}}
\]

for \( |y| < 2 \) and zero otherwise.
As in (5-43) \[ F_y[g(x)] = F_x(x) \]

\[ \frac{1 + x}{2} = 1 - e^{-2y} \quad y = g(x) = -\frac{1}{2} \ln \frac{1 - x}{2} \]

for \( |x| < 1 \). For \( x < -1 \), \( g(x) = 0 \); for \( x \geq 1 \), \( g(x) = \infty \).

5-14 (a) \[ g(x) = 2F_x(x) + 4 \quad g'(x) = 2f_x(x) \]

If \( 4 < y < 6 \) then \( y = 2F_x(x) + 4 \) has a unique solution \( x_1 \) and

\[
\frac{f_y(y)}{2f_x(x_1)} = \frac{1}{2} \quad \frac{4}{2} \]

(b) Similarly \( g(x) = 2F_x(x) + 8 \)

5-15 (a) The RV \( x \) takes the values \( k = 0, 1, \ldots, 10 \) and

\[ P(x = k) = p_k = \binom{10}{k} \frac{1}{2^{10}} \quad 0 \leq k \leq 10 \]

\( F_x(x) \) is a staircase function with discontinuities at the points \( x = k \) and jumps equal to \( p_k \).

(b) The RV \( y = (x - 3)^2 \) takes the values \( y = k^2 \) for \( k = 0, 1, \ldots, 7 \) and

<table>
<thead>
<tr>
<th>( k )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
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<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>( q_k )</td>
<td>( p_3 )</td>
<td>( p_2 + p_4 )</td>
<td>( p_1 + p_5 )</td>
<td>( p_0 + p_6 )</td>
<td>( p_7 )</td>
<td>( p_8 )</td>
<td>( p_9 )</td>
<td>( p_{10} )</td>
</tr>
</tbody>
</table>
5.16 

\[ X \sim \text{Beta}(\alpha, \beta) \quad \text{gives} \]

\[ f_X(x) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1} (1 - x)^{\beta-1}, \quad 0 < x < 1. \]

\[ Y = 1 - X \quad \Rightarrow \quad x_1 = 1 - y, \quad \left| \frac{dy}{dx} \right| = 1 \]

\[ \Rightarrow F_Y(y) = \frac{1}{\left| \frac{dy}{dx} \right|} f_X(1 - y) = \begin{cases} \frac{1}{B(\beta, \alpha)} y^{\beta-1} (1 - y)^{\alpha-1}, & 0 < y < 1 \\ 0, & \text{otherwise.} \end{cases} \]

This gives

\[ Y \sim \text{Beta}(\beta, \alpha). \]

5.17

\[ X \sim \chi^2(n) \Rightarrow \]

\[ f_X(x) = \frac{1}{2^{n/2} \Gamma(n/2)} x^{n/2 - 1} e^{-x/2} U(x) \]

\[ y = \sqrt{x} \quad \Rightarrow \quad x_1 = y^2 \]

\[ \frac{dy}{dx} = \frac{1}{2y} \]

Thus

\[ f_Y(y) = 2y f_X(y^2) = \frac{y^{n-1}}{2^{n/2 - 1} \Gamma(n/2)} e^{-y^2/2} U(y) \]

and it represents the chi-distribution.

5.18

\[ X \sim U(0, 1) \]

\[ Y = -2 \log X \quad \Rightarrow \quad x_1 = e^{-y/2} \]

\[ \frac{dy}{dx} = -\frac{2}{x} = -2e^{y/2} \]

\[ f_Y(y) = \frac{1}{\left| \frac{dy}{dx} \right|} f_X(x_1) = \frac{1}{2} e^{-y/2} U(y) \]

\[ \sim \text{Exponential}(2) \equiv \chi^2(2) \]

5.19

\[ f_X(x) = \lambda e^{-\lambda x} u(x) \]

\[ Y = X^{1/\beta} \quad \Rightarrow \quad x_1 = y^\beta \]

\[ \left| \frac{dy}{dx} \right| = \frac{1}{\beta} x^{1/\beta - 1} = \frac{1}{\beta} y^{1-\beta} \]

\[ f_Y(y) = \frac{1}{\left| \frac{dy}{dx} \right|} f_X(x_1) = \lambda \beta y^{\beta-1} e^{-\lambda y^\beta} U(y) \]

and it represents Weibull distribution.
5-20 For \(|y| < a\) the equation \(y = a \sin \omega t\) has infinitely many solutions \(\tau_i\); in each interval of length \(2\pi/\omega\) there are two such solutions. Furthermore, 

\[ y'(t) = \omega \sqrt{a^2 - y^2} \]

\[ \tau_i = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin \theta}{a} \] \(\tau_{i+2} - \tau_i \xrightarrow{\omega \to \infty} 0 \)

Hence,

\[ \frac{1}{\omega \sqrt{a^2 - y^2}} \int_{-\infty}^{\infty} f_t(\tau_i) \xrightarrow{\omega \to \infty} \frac{1}{\sqrt{a^2 - y^2}} \int_{-\infty}^{\infty} f_t(\tau) \, d\tau = \frac{1}{\pi \sqrt{a^2 - y^2}} \]

5-21 If \(y > 0\) then

\[ F_y(y|x > 0) = F_x(\sqrt{y}|x > 0) + F_x(-\sqrt{y}|x > 0) = F_x(\sqrt{y}|x > 0) \]

\[ F_x(\sqrt{y}|x > 0) = \frac{P(0 < x < \sqrt{y})}{P(x > 0)} = \frac{F_x(\sqrt{y}) - F_x(0)}{1 - F_x(0)} \]

\[ f_y(y|x > 0) = \frac{d}{dy} F_y(\sqrt{y}|x > 0) = \frac{f_x(\sqrt{y})}{2\sqrt{y}[1-F_x(0)]} \]

5-22 (a) \(\eta_y = a \eta_x + b\)

\[ \sigma_y^2 = E\{(a \eta_x + b - (a \eta_x + b))^2\} \]

\[ \sigma_y^2 = E(a^2(x - \eta_x)^2) = a^2 \sigma_x^2 \]

(b) \(y = \frac{x - \eta_x}{\sigma_x}\)

\[ E(y) = 0 \quad \sigma_y^2 = \frac{\sigma_x^2}{\sigma_x^2} = 1 \]

5-23 If \(x\) has a Rayleigh density, then [see (5-76)]

\[ E(x^2) = 2\alpha^2 \quad E(x^4) = 8\alpha^4 \]

If \(y = b + cx^2\), then

\[ E(y) = b + 2\alpha^2 c \quad E(y^2) = b^2 + 4\alpha^4 c^2 + 8\alpha^4 c^2 \]

\[ \sigma_y^2 = E(y^2) - E_y^2 = 4\alpha^4 c^2 \]
5 - 24 \[ y = 3x^2 \quad E(x^2) = \sigma_x^2 = 4 \quad E(x^4) = 3\sigma_x^4 = 48 \]

\[ E(y) = 12 \quad E(y^2) = 9 \times 48 = 432 \quad \sigma_y^2 = 432 - 144 = 288 \]

If \( y > 0 \) then \( 3x^2 = y \) for \( x = \pm \sqrt[3]{y/3} \)

\[ y' = 6x \]

\[ f_y(y) = \frac{24}{\sqrt{12y}} \quad f_x(\sqrt[3]{y}/3) = \frac{1}{\sqrt{2\pi\sqrt{y}}} e^{-y/24} u(y) \]

---

5.25

\( X \sim B(n, p) \Rightarrow P(X = k) = \binom{n}{k} p^k q^{n-k}, \quad k = 0, 1, 2, \ldots n. \)

a)

\[ E(X) = \sum_{k=0}^{n} k P(X = k) = \sum_{k=1}^{n} k \frac{n!}{k!(n-k)!} p^k q^{n-k} \]

\[ = np \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!(n-k)!} p^{k-1} q^{n-k} \]

\[ = np(p + q)^{n-1} = np. \]

b)

\[ E[X(X-1)] = \sum_{k=2}^{n} k(k-1) \frac{n!}{k!(n-k)!} p^k q^{n-k} \]

\[ = n(n-1)p^2 \sum_{k=2}^{n} \frac{(n-2)!}{(k-2)!(n-k)!} p^{k-2} q^{n-k} \]

\[ = n(n-1)p^2 (p + q)^{n-2} \]

\[ = n(n-1)p^2 \]

c)

\[ E[X(X-1)(X-2)] = \sum_{k=3}^{n} k(k-1)(k-2) \frac{n!}{k!(n-k)!} p^k q^{n-k} \]

\[ = n(n-1)(n-2)p^3 \sum_{k=3}^{n} \frac{(n-3)!}{(k-3)!(n-k)!} p^{k-3} q^{n-k} \]

\[ = n(n-1)(n-2)p^3 (p + q)^{n-3} \]

\[ = n(n-1)(n-2)p^3 \]

\[ E(X^2) = E(X(X-1)) + E(X) = n^2 p^2 + npq \]

\[ E(X^3) = E(X(X-1)(X-2)) + 3E(X^2) - 2E(X) \]

\[ = n(n-1)(n-2)p^3 + 3(n^2 p^2 + npq) - 2np \]

\[ = n^3 p^3 + 3n^2 p^2 q + npq(q - p). \]
\[ X \sim P(\lambda) \quad \Rightarrow \quad P(X = k) = e^{\lambda} \frac{\lambda^k}{k!}, \quad k = 0, 1, 2, \ldots \]

a) \[ E(X) = \lambda, \quad \text{Var}(X) = \sigma_X^2 = \lambda \]

From Chebyshev’s inequality (5-88)

\[ P(|X - \mu| < \lambda) > 1 - \frac{\sigma^2}{\lambda^2} = 1 - \frac{1}{\lambda} \]

But \[ |X - \mu| < \lambda = |X - \lambda| < \lambda \quad \Rightarrow \quad 0 < X < 2\lambda \]

which gives

\[ P(0 < X < 2\lambda) > 1 - \frac{1}{\lambda} = \frac{\lambda - 1}{\lambda}. \]

b) \[ E[X(X - 1)] = \sum_{k=2}^{\infty} k(k - 1) e^{-\lambda} \frac{\lambda^k}{k!} \]

\[ = e^{-\lambda} \lambda^2 \sum_{k=2}^{\infty} \frac{\lambda^{k-2}}{(k-2)!} = e^{-\lambda} \lambda^2 e^\lambda = \lambda^2. \]

\[ E[X(X - 1)(X - 2)] = \sum_{k=3}^{\infty} k(k - 1)(k - 2) e^{-\lambda} \frac{\lambda^k}{k!} \]

\[ = e^{-\lambda} \lambda^3 \sum_{k=3}^{\infty} \frac{\lambda^{k-3}}{(k-3)!} = \lambda^3. \]

5-27 Follows from (4-74)

\[ E\{x\} = \int_{-\infty}^{\infty} xf(x)\,dx = \int_{-\infty}^{\infty} x \sum_{A_1 = \perp} f(x|A_1)P(A_1)\,dx \]

because \[ E\{x|A_1\} = \int_{-\infty}^{\infty} xf(x|A_1)\,dx \]

5-28 From (5-89) with \( a = \sqrt{n} \):

\[ P(\chi \geq \sqrt{n}) \leq n/\sqrt{n} = \sqrt{n} \]
5-29 From (5-86) with \( g(x) = x^3 \) \( g''(x) = 6x \):

\[
E(x^3) = \eta^3 + 6\eta \frac{\sigma^2}{2} = 1120
\]

5-30 (a) If \( y = x^3 \), then \( x = \sqrt[3]{y} \) \( g'(x) = 3x^2 = \frac{3}{\sqrt[3]{y}} \)

But \( f_x(x) = 0.5 \) for \( 10 < x < 12 \), i.e., for \( 10^3 < y < 12^3 \)

and (5-16) yields

\[
f_y(y) = \frac{0.5}{\frac{3}{\sqrt[3]{y}}} \quad 10^3 < y < 12^3
\]

and zero otherwise.

(b) 1.

\[
E(x^3) = 0.5 \int_{10}^{12} x^3 \, dx = 1342
\]

2. With \( g(x) = x^3 \) \( E(x) = 11 \) \( \sigma_x^2 = 1/3 \), (5-86) yields

\[
E(x^3) \approx 11^3 + 6 \times 11 \times \frac{1}{6} \approx 1342
\]

5-31 With \( g(x)=1/x \), \( g''(x)=2/x^3 \), \( \eta=100 \), and \( \sigma=3 \), (5-55) yields

\[
E \left\{ \frac{1}{x} \right\} \approx \frac{1}{100} + \frac{9}{2} \times \frac{2}{100^3} = 0.010009
\]
\[
\frac{\partial |x-a|}{\partial a} = \begin{cases} 
1 & x < a \\
-1 & x > a 
\end{cases}
\quad \text{If } I(a) = E(|x-a|) \text{ then}
\]
\[
dI(a) = \frac{\partial |x-a|}{\partial a} = 1 \mathbb{P}(x < a) - 1 \mathbb{P}(x > a)
\]
\[
= 2F(a) - 1
\]

(a)
\[
I(a) = I(m) + \int_{m}^{a} I'(x)dx = I(m) + \int_{m}^{a} \left(2F(a) - 1\right)dx
\]
\[
= E(|x - m|) - 2 \int_{m}^{a} x f(x)dx
\]

because
\[
\int_{m}^{a} F(a)dx = aF(a) - mF(m) - \int_{m}^{a} xf(x)dx
\]
\[
F(m) = \frac{1}{2} \int_{m}^{a} f(x)dx = F(a) - F(m)
\]

(b) \(I(a) = E(|x - a|)\) is minimum if
\[
I'(a) = 2F(a) - 1 = 0 \quad \text{i.e. if } F(a) = \frac{1}{2} \quad a = m
\]

5-33
\[
E(|x|) = \int_{-\infty}^{\infty} xf(x)dx - \int_{0}^{\infty} xf(x)dx
\]
\[
\eta = E(x) = \int_{0}^{\infty} xf(x)dx + \int_{-\infty}^{0} xf(x)dx
\]
\[
E(\frac{|x| + \eta}{2}) = \int_{0}^{\infty} xf(x)dx = \frac{1}{\sigma\sqrt{2\pi}} \int_{0}^{\infty} x e^{-\frac{(x-\eta)^2}{2\sigma^2}}dx
\]
\[
- \frac{1}{\sigma \sqrt{2\pi}} \int_0^{\infty} (x+\eta) e^{-\frac{(x-\eta)^2}{2\sigma^2}} \, dx = \frac{1}{\sigma \sqrt{2\pi}} \int_\eta^{\infty} e^{-\frac{y^2}{2\sigma^2}} \, dy = \frac{\sigma}{\sqrt{2\pi}} e^{-\frac{\eta^2}{2\sigma^2}}
\]
\[
\frac{1}{\sigma \sqrt{2\pi}} \int_0^{\infty} e^{-\frac{(x-\eta)^2}{2\sigma^2}} \, dx = G(\frac{\eta}{\sigma})
\]

Multiplying the last line by \( \eta \) and subtracting from the fourth line, we obtain
\[
\frac{E(|x|+\eta)}{2} = \frac{\sigma}{\sqrt{2\pi}} e^{-\frac{\eta^2}{2\sigma^2}} + G(\frac{\eta}{\sigma})
\]

5-34 The proof is given in sec 14-3: [see (14-100)].

5-35 (a) Follows from (5-89)  (b) \( e^{sx} \geq e^{sA} \) iff \( x \geq A \) for \( s > 0 \) and \( x \leq A \) for \( s < 0 \).

5.36 See proof for Lyapunov inequality (Ch.5, Eq.(5-92).)
5-37 (a) If \( \phi(\omega) = e^{-\alpha|\omega|} \) then [see (5-102)]

\[
\phi(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\alpha|\omega|} e^{j\omega x} d\omega = \frac{1}{\pi} \int_{0}^{\infty} e^{-\alpha u} u^e^{-\alpha u} d\omega = \frac{\alpha}{\pi(\alpha^2 + x^2)}
\]

(b) If \( f(x) = \frac{\alpha}{2} e^{-\alpha|x|} \), then [see (5-94)]

\[
\phi(\omega) = \frac{\alpha}{2} \int_{-\infty}^{\infty} e^{-\alpha|x|} e^{-j\omega x} dx = \alpha \int_{0}^{\infty} e^{-\alpha x} \cos \omega x dx = \frac{\alpha^2}{\alpha^2 + \omega^2}
\]

5.38 a) On comparing Eq.(4-34) with Eq.(5-106), Example 5-29, we get

\[ X \sim G(\alpha, \beta) \Rightarrow \phi_X(\omega) = (1 - j\beta \omega)^{-\alpha} \]

\[ \phi'_X(\omega) = -\alpha(1 - j\beta \omega)^{-\alpha+1} (-j\beta) \]

so that

\[ E(X) = \frac{1}{j} \phi'_X(0) = \alpha \beta. \]

Similarly

\[ \phi''_X(\omega) = j\alpha \beta (\alpha + 1) (1 - j\beta \omega)^{-\alpha+2} (j\beta) \]

and hence

\[ E(X^2) = \frac{1}{j^2} \phi''_X(0) = \alpha \beta^2 (\alpha + 1). \]

Thus

\[ \text{Var}(X) = E(X^2) - (E(X))^2 = \alpha \beta^2. \]

b) \( X \sim \chi^2(n) \Rightarrow \alpha = \frac{n}{2}, \beta = 2 \)

in Gamma(\(\alpha, \beta\)). This gives

\[ \phi_X(\omega) = (1 - j2\omega)^{-n/2} \]

\[ E(X) = n \]

\[ \text{Var}(X) = 2n. \]

c) \( X \sim B(n, p) \).

From Prob 5-25 (a)-(b)

\[ E(X) = np \]

\[ \text{Var}(X) = E(X(X-1)) + E(X) = npq. \]

and

\[ \phi_X(\omega) = \sum_{k=0}^{n} e^{jk\omega} P(X = k) \]

\[ = \sum_{k=0}^{n} \binom{n}{k} (pe^{j\omega})^k q^{n-k} = (pe^{j\omega} + q)^n. \]
d) 

\[ X \sim N \text{ Binomial} (r, p). \]

From (4-64)

\[
\phi_X(\omega) = \sum_{k=0}^{\infty} e^{ik\omega} P(X = k) \\
= \sum_{k=0}^{\infty} \binom{r}{k} \frac{k!}{(r-k)!} \left( pe^{i\omega} \right)^k \\
= p^r \sum_{k=0}^{\infty} \binom{r}{k} \left( -qe^{i\omega} \right)^k \\
= p^r (1 - qe^{i\omega})^{-r}.
\]

\[
\gamma(z) = \sum_{k=0}^{\infty} p^k q^{k-1} z^k = \frac{p}{1 - qz} \quad q = 1-p
\]

\[
\gamma'(z) = \frac{p q}{(1-qz)^2} \\
\gamma'(1) = \frac{p q}{(1-q)^2} = \frac{p}{q} = \eta_x
\]

\[
\gamma''(z) = \frac{2 p q^2}{(1-qz)^3} \\
\gamma''(1) = \frac{2 q^2}{p^2} = m_2 - m_1
\]

\[
\sigma^2 = m_2 - m_1 = \frac{2 q^2}{p^2} + m_1 - m_1 = \frac{q}{p}
\]

\[ \[ \]

5-40

\[
\gamma(z) = p^n \sum_{k=0}^{\infty} \binom{n}{k} (-q)^k z^k = p^n (1-qz)^{-n}
\]

(binomial expansion with negative exponent)

\[
\gamma'(z) = \frac{n p^n q}{(1-qz)^{n+1}} \\
\gamma'(1) = \frac{n q}{p} = \eta_x
\]

\[
\gamma''(z) = \frac{n(n+1) p^n q^2}{(1-qz)^{n+2}} \\
\gamma''(1) = \frac{n(n+1) q^2}{p^2} = m_2 - m_1
\]

\[
\sigma_x^2 = \gamma''(1) + m_1 - m_1 = \frac{n q}{p^2}
\]

39
5.41 We have

\[ P(X = k) = \binom{k-1}{r-1} p^r q^{k-r}, \quad k = r, r+1, \ldots \]

Let \( k = n + r \) so that

\[ P(X = n + r) = \binom{n+r-1}{r-1} p^r q^n, \quad n = 0, 1, 2, \ldots \]

\[ = \frac{(n+r-1)!}{n!(r-1)!} p^r (1-p)^n \]

\[ = \frac{1}{n!} \frac{(n+r-1)(n+r-2)\cdots(r)}{r^n} [r(1-p)]^n p^r \]

\[ = \frac{\lambda^n}{n!} \left\{ \left(1 + \frac{n-1}{r} \right) \left(1 + \frac{n-2}{r} \right) \cdots \right\} (1 - \frac{r(1-p)}{r})^r \]

\[ = \frac{\lambda^n}{n!} \left\{ \prod_{k=1}^{n} \left(1 + \frac{n-k}{r} \right) \right\} \left(1 - \frac{\lambda}{r} \right)^r, \]

where \( \lambda = r(1-p) \). Thus

\[ \lim_{r \to \infty} P(X = n + r) = \frac{\lambda^n}{n!} \left\{ \lim_{r \to \infty} \prod_{k=1}^{n} \left(1 + \frac{n-k}{r} \right) \right\} \lim_{r \to \infty} \left(1 - \frac{\lambda}{r} \right)^r \]

\[ \to \frac{\lambda^n}{n!} e^{-\lambda} \sim P(\lambda). \]
5-42 \[ E(e^{sX}) = e^{sn} E(e^{s(x-n)}) = e^{sn} E\left\{ \sum_{n=0}^{\infty} \frac{s^n}{n!} (x-n)^n \right\} \]

\[ = e^{sn} \sum_{n=0}^{\infty} \frac{s^n}{n!} \mu_n \]

5-43 If \( \Phi(\omega_1) = 0 \), then [see also (9-176)]

\[ \int_{-\infty}^{\infty} (1-e^{j\omega_1 x})f(x)dx = 0, \text{ hence, } f(x) = \sum_{n=0}^{\infty} p_n \delta(x - \frac{2\pi n}{\omega_1}) \]

5-44 (a) If \( \eta = 0 \), then \( m_n = \mu_n \), \( \lambda_1 = \eta = 0 \)

\[ \phi(s) = \sum_{n=0}^{\infty} \frac{\mu_n}{n!} s^n \quad \psi(s) = \sum_{n=2}^{\infty} \frac{\lambda_n}{n!} s^n \]

\[ 1 + \frac{\mu_2}{2!} s^2 + \frac{\mu_3}{3!} s^3 + \frac{\mu_4}{4!} s^4 + \cdots = \exp\left\{ \frac{\lambda_2}{2!} s^2 + \frac{\lambda_3}{3!} s^3 + \frac{\lambda_4}{4!} s^4 + \cdots \right\} \]

Expanding the exponential and equating powers of \( s \), we obtain

\[ \mu_2 = \lambda_2 \quad \mu_3 = \lambda_3 \quad \frac{\mu_4}{4!} = \frac{\lambda_4}{4!} + \frac{1}{2!} \left( \frac{\lambda_2}{2!} \right)^2 \]

(b) If \( y \) is \( N(0; \sigma_y^2) \) then

\[ \varphi_y(s) = \frac{\lambda_2}{2} s^2 \], hence, \( \lambda_n = 0 \) for \( n \geq 3 \)
\[ P(y = 0) = P(x \leq 1) = p_0 + p_1 \]

\[ P(y = k) = P(x = k + 1) = p_{k+1} \quad k \geq 1 \]

\[ \Gamma_y(z) = p_0 + p_1 + \sum_{k=1}^{\infty} p_{k+1} z^k = p_0 + z^{-1}[\Gamma_x(z) - p_0] \]

\[ \eta_y = \sum_{k=1}^{\infty} k p_{k+1} = \sum_{r=1}^{\infty} r p_r - \sum_{r=1}^{\infty} p_r = \eta_x - 1 + p_0 \]

\[ E(y^2) = \sum_{k=1}^{\infty} k^2 p_{k+1} = \sum_{r=1}^{\infty} (r-1)^2 p_r = E(y^2) - 2\eta_x + 1 - p_0 \]

\[ 0 \leq E \left\{ \left| \sum_{i=1}^{n} a_i e^{j\omega_i x} \right|^2 \right\} = E \left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} a_i^* a_j e^{j(\omega_i - \omega_j) x} \right\} \]

\[ = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i^* a_j \phi(\omega_i - \omega_j) \]

5-47 From the assumptions it follows that

\[ g'(x) = -g''(x) \quad g''(x) \geq 0 \quad f(x-n) = f(n-x) \]

Hence, if \( I(a) = E(g(x-a)) \), then

\[ I'(a) = - \int_{-\infty}^{\infty} g'(x-a)f(x)dx \quad I'(n) = 0 \]

\[ I''(a) = \int_{-\infty}^{\infty} g''(x-a)f(x)dx \geq 0 \quad \text{all } a \]

Hence, \( I(a) \) is minimum for \( a = n \).
\[ f(x, \nu) = \frac{1}{\sqrt{2\pi \nu}} e^{-x^2/2\nu} \]

\[ \frac{\partial f}{\partial \nu} = \frac{-1 + x^2/\nu}{2\nu \sqrt{\nu}} e^{-x^2/2\nu} \]

\[ \frac{\partial^2 f}{\partial x^2} = \frac{-1 + x^2/\nu}{\nu \sqrt{\nu}} e^{-x^2/2\nu} \]

Hence

\[ \frac{\partial f}{\partial \nu} = \frac{1}{2} \frac{\partial^2 f}{\partial x^2} \tag{1} \]

(see also (6-198) - (6-199))

(a) Integrating by parts, using (1) and assuming that \( g^{(k)}(x)f(x) \to 0 \) as \( |x| \to \infty \), \( k = 0, 1, 2 \), we obtain

\[ E(g''(x)) = \int_{-\infty}^{\infty} \frac{d^2 g}{dx^2} f dx = \int_{-\infty}^{\infty} g \frac{\partial^2 f}{\partial x^2} dx = 2 \int_{-\infty}^{\infty} g \frac{\partial f}{\partial \nu} dx \]

\[ = 2 \frac{d}{d\nu} \int_{-\infty}^{\infty} g f dx = 2 \frac{d}{d\nu} E(g(x)) \]

(b) The moments \( \mu_n(\nu) = E(x^n) \) of \( x \) depend on the variance \( \nu \) of \( x \) and (i) yields

\[ \mu'(\nu) = \frac{d}{d\nu} E(x^n) = \frac{1}{2} E(n(n-1)x^{n-2}) = \frac{n(n-1)}{2} \mu_{n-2}(\nu) \]

Furthermore, \( \mu_n(0) = 0 \) because, if \( \nu = 0 \), then \( x = 0 \).

Hence

\[ \mu_n(\nu) = \frac{n(n-1)}{2} \int_{0}^{\nu} \mu_{n-2}(\beta) d\beta \]

43
The function

\[ r(e^{j\omega}) = \mathbb{E}(e^{jx\omega}) = \sum_{k=0}^{\infty} p_k e^{jkw} \]

is periodic with period \(2\pi\) and Fourier series coefficients \(p_k = \mathbb{E}(x = k)\).

5.50 The event \(\{X = 1\}\) is given by the disjoint union "TH ∪ HT". Similarly, the event "X = k" is given by the union of the disjoint events (k "T"s followed by "H" or k "H"s followed by "T")

"TT ⋯ TTH" ∪ "HH ⋯ HHT", \(k = 1, 2, \cdots\)

Thus

\[ P(X = k) = P("TT ⋯ TTH" ∪ "HH ⋯ HHT") = P(TT ⋯ TH) + P(HH ⋯ HT) = q^k p + p^k q, \quad k = 1, 2, \cdots \]

Also

\[ E(X) = \sum_{k=1}^{\infty} kP(X = k) \]

\[ = \sum_{k=1}^{\infty} kq^k p + \sum_{k=1}^{\infty} kp^k q = pq \left\{ \sum_{k=1}^{\infty} kq^{k-1} + \sum_{k=1}^{\infty} kp^{k-1} \right\} \]

\[ = pq \left\{ \frac{\partial}{\partial q} \sum_{k=1}^{\infty} q^k + \frac{\partial}{\partial p} \sum_{k=1}^{\infty} p^k \right\} = pq \left\{ \frac{\partial}{\partial q} \left( \frac{q}{1-q} \right) + \frac{\partial}{\partial p} \left( \frac{p}{1-p} \right) \right\} \]

\[ = pq \left\{ \frac{1}{p^2} + \frac{1}{q^2} \right\} = \frac{p}{q} + \frac{q}{p}. \]
5.51 (a) When samples are drawn with replacement, probability of each item being defective is given by

\[ p = \frac{M}{N} < 1 \quad \text{(constant)} \]

and

\[ q = 1 - p = \frac{N - M}{M} < 1 \]

represents the constant probability that the chosen item is not defective. In that case (with replacement), there are \( \binom{n}{k} \) possible ways of arranging \( k \) defective items among \( n \) chosen items, and each such arrangement has probability \( p^k q^{n-k} \). This gives

\[ P(X = k) = \binom{n}{k} p^k q^{n-k}, \quad k = 0, 1, 2, \ldots n \]

which represents the Binomial distribution.

(b) If the samples are drawn without replacement, there are \( \binom{M}{k} \) possible ways of choosing \( k \) defective item from a total of \( M \) defective items, and \( \binom{N-M}{n-k} \) possible ways of choosing \( n-k \) “good” items from \( (N-M) \) “good” items independently. This gives

\[ \binom{M}{k} \binom{N-M}{n-k} \]

to be the total number of ways of selecting \( k \) defective items and \( n-k \) “good” items from a subsample of \( M \) and \( N-M \) items respectively (favorable ways). But there are a total of \( \binom{N}{n} \) ways of selecting \( n \) items among \( N \) items. This gives

\[ P(X = k) = \frac{\binom{M}{k} \binom{N-M}{n-k}}{\binom{N}{n}}, \]

since \( 0 \leq k \leq M \) and \( n-k \leq N-M, n-k \geq 0 \), i.e. \( 0 \leq k \leq M, k \leq n, k \geq n + M - N \).

(c) From (b)

\[ P(X = k) = \frac{M!}{k!(M-k)!} \frac{(N-M)!}{(n-k)!(N-M-n+k)!} \frac{n!(N-n)!}{N!} \]

\[ = \binom{n}{k} \frac{M(M-1) \cdots (M-k+1)}{N(N-1) \cdots (N-k+1)} \frac{(N-M)(N-M-1) \cdots (N-M-n+k+1)}{(N-k)(N-k-1) \cdots (N-n+1)} \]

\[ \approx \binom{n}{k} \frac{M}{N}^k \left( \frac{N-M}{N} \right)^{n-k} = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, 2, \ldots n \]

since \( N \to \infty, M \to \infty \) such that \( M/N \to p \), and \( n \ll N \). Thus

\[ P(X = k) \to \text{Binomial}(n, p = M/N) \]

under the above conditions.
5.52 (a) Refer to discussions in problem 5.51 (a) if sampling is done with replacement, then

\[ p = \frac{n}{n+m} \]

represents the probability of selecting a white marble on any trial. The event \( "X = k" \) is given by \( "r - 1 \) white mables among the first \( k - 1 \) trials" followed by \( "a white marble at the \( k^{th} \) trial". But from problem 5.51 (a), the event \( r - 1 \) white mables among the first \( k - 1 \) trials has a binomial distribution whose probability is given by \( \binom{k-1}{r-1} p^{r-1} q^{k-r} \). Thus

\[ P(X = k) = \binom{k-1}{r-1} p^{r-1} q^{k-r} p = \binom{k-1}{r-1} p^r q^{k-r}, \quad k = r, r+1, \ldots \]

which represents the Negative-binomial distribution.

(b) If sampling is done with replacement, then the favorable ways of choosing the white balls are given by:

(i) \( \binom{k-1}{r-1} \) ways of selecting \( r - 1 \) white balls among the first \( k - 1 \) trials/balls.

(ii) One ways of selecting (the \( r^{th} \)) white ball at the \( k^{th} \) trial

(iii) \( \binom{m+n-k}{n-r} \) ways of selecting the remaining \( n - r \) white balls among the remaining \( m + n - k \) balls.

This gives \( \binom{k-1}{r-1} \cdot 1 \cdot \binom{m-n-k}{n-r} \) to be the total number of favorable ways of selecting the white balls. Since there are \( n+m \) balls there are a total of \( \binom{n+m}{n} \) ways of selecting \( n \) white balls. This gives

\[ P(X = k) = \binom{k-1}{r-1} \frac{\binom{m+n-k}{n-r}}{\binom{n+m}{n}}, \quad k = r, r+1, \ldots \]

(c) From (b)

\[ P(X = k) = \binom{k-1}{r-1} \frac{(m+n-k)!}{(n-r)! (m-k+r)! (m+n)!} \]

\[ = \binom{k-1}{r-1} \frac{(m+n)}{(m+n-1)} \frac{(n-1)}{(m+n-1)} \cdots \frac{(n+r-1)}{(m+n-r+1)} \frac{m!(m+n-k)!}{(m-n-r)! (m-k+r)!} \]

\[ \approx \binom{k-1}{r-1} \frac{(m+n)^r}{(m+n-r)^r} \frac{(m-n-r)}{(m+n-r)} \cdots \frac{(m-k+r+1)}{(m+n-k+1)} \]

\[ \approx \binom{k-1}{r-1} \left( \frac{m+n}{m+n-r} \right)^{k-r}, \quad k = r, r+1, \ldots, \quad q = 1-p \]

\[ \sim NB \left( r, \frac{p=n/(n+m)}{ } \right). \]
6.1 (a) Define

\[ Z = X + Y \]

Note that both \( X \) and \( Y \) positive random variables hence (use Eq. (6-45))

\[
f_Z(z) = \int_0^z f_{XY}(z-y, y) \, dy = \int_0^z e^{-(z-y)+y} \, dy = z \, e^{-z} U(z).
\]

(b)

\[ Z = X - Y \]

\( Z \) ranges over the entire real axis for the random variables \( X \) and \( Y \) (see Eq. (6-55))

\[
F_Z(z) = \begin{cases} 
\int_0^\infty \int_0^{z+y} f_{XY}(x, y) \, dx \, dy, & z > 0 \\
\int_{-\infty}^\infty \int_0^{z+y} f_{XY}(x, y) \, dx \, dy, & z < 0
\end{cases}
\]

Differentiation gives

\[
f_Z(z) = \begin{cases} 
\int_0^\infty f_{XY}(z+y, y) \, dy, & z > 0 \\
\int_{-\infty}^\infty f_{XY}(z+y, y) \, dy, & z < 0
\end{cases}
\]

\[
f_Z(z) = \begin{cases} 
\int_0^\infty e^{-(z+y)+y} \, dy = e^{-z} \int_0^\infty e^{-2y} \, dy = \frac{1}{2} e^{-z}, & z > 0 \\
\int_{-\infty}^\infty e^{-(z+y)+y} \, dy = e^{-z} \int_{-\infty}^\infty e^{-2y} \, dy = \frac{1}{2} e^{z}, & z < 0
\end{cases}
\]

or

\[
f_Z(z) = \frac{1}{2} e^{-|z|}, \quad -\infty \leq z \leq \infty.
\]

(c)

\[ Z = XY. \]

\[
F_Z(z) = P\{Z \leq z\} = P\{XY \leq z\} = \int_0^\infty \int_0^{z/y} f_{XY}(x, y) \, dx \, dy
\]

or (see Eq. (6-148))

\[
f_Z(z) = \int_0^\infty \frac{1}{y} f_{XY}(\frac{z}{y}, y) \, dy = \int_0^\infty \frac{1}{y} e^{-(z/y)+y} \, dy
\]
(d) 
\[ Z = X/Y \]
\[ F_Z(z) = P\{Z \leq z\} = P\{ \frac{X}{Y} \leq z\} \]
\[ = \int_0^\infty \int_0^\infty f_{XY}(x,y)dx\,dy \]

(use Eq. (6-60))
\[ f_Z(z) = \int_0^\infty y f_{XY}(yz,y)dy = \int_0^\infty y e^{y(z+1)}dy = \int_0^\infty y e^{(1+z)y} \]
\[ = \left[ \frac{y e^{-(1+z)y}}{-(1+z)} \right]_0^\infty + \left( \frac{1}{1+z} \right) \int_0^\infty e^{(1+z)y}dy \]
\[ = \left( \frac{1}{1+z} \right) \left[ \frac{e^{-(1+z)y}}{-(1+z)} \right]_0^\infty = \frac{1}{(1+z)^2} U(z) \]

(e)
\[ Z = \min (X,Y) \]
\[ F_Z(z) = P\{\min (X,Y) \leq z\} \]
\[ = 1 - P\{Z > z, Y > z\} \]
\[ = 1 - [1 - F_X(z)] [1 - F_Y(z)] \]
\[ = F_X(z) + F_Y(z) - F_X(z) F_Y(z) \]

(see Eq. (6-81))
\[ f_Z(z) = f_X(z) + f_Y(z) - F_X(z)f_Y(z) - f_X(z)F_Y(z). \]

We have
\[ f_X(z) = f_Y(z) = e^{-z} U(z) \]
so that
\[ F_X(z) = \int_0^z e^{-x}dx = (1 - e^{-z}) U(z) = F_Y(z) \]
\[ f_Z(z) = [e^{-z} + e^{-z} - 2(1 - e^{-z}) e^{-z}] U(z) \]
\[ = 2e^{-z} [1 - 1 + e^{-z}] U(z) \]
\[ = 2e^{-2z} U(z) \sim \text{Exponential (2)}. \]

(f)
\[ Z = \max(X,Y) \]
\[ F_Z(z) = P\{\max(X,Y) \leq z\} = P\{X \leq z, Y \leq z\} \]
\[ = P\{X \leq z\} P\{Y \leq z\} = F_X(z) F_Y(z) \]
\[ f_Z(z) = F_X(z) f_Y(z) + f_X(z) F_Y(z) \]
\[ = e^{-z} (1 - e^{-z}) + e^{-z} (1 - e^{-z}) \]
\[ = 2e^{-z} (1 - e^{-z}) U(z) \]

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\( Z = \frac{\min(X, Y)}{\max(X, Y)} \), \( 0 < z < 1 \)

\[
F_Z(z) = P \left\{ \left( \frac{\min(X, Y)}{\max(X, Y)} \right) \leq z \right\} \cap ((X \leq Y) \cup (X > Y))
\]

\[
= P \left\{ \left( \frac{\min(X, Y)}{\max(X, Y)} \right) \leq z \right\} \cap (X \leq Y)
+ P \left\{ \left( \frac{\min(X, Y)}{\max(X, Y)} \leq z \right) \cap (X > Y) \right\}
\]

\[
= P\{X/Y \leq z, X \leq Y\} + P\{Y/X \leq z, X > Y\}
= P\{X \leq Y \cap X \leq z\} + P\{Y \leq X \cap X > z\}
\]

\[
= \int_0^\infty \int_0^{yz} f_{XY}(x, y) \, dx \, dy + \int_0^\infty \int_0^{xz} f_{XY}(x, y) \, dy \, dx
\]

\[
f_Z(z) = \int_0^\infty y f_{XY}(yz, y) \, dy + \int_0^\infty x f_{XY}(x, xz) \, dx
\]

\[
= \int_0^\infty y f_{XY}(yz, y) \, dy + \int_0^\infty y f_{XY}(y, yz) \, dy
\]

\[
= \int_0^\infty y \left( e^{-(yz+y)} + e^{-(y+yz)} \right) \, dy
\]

\[
= 2 \int_0^\infty ye^{-y(1+z)} \, dz = \begin{cases} \frac{2}{(1+z)^2}, & 0 \leq z \leq 1 \\ 0, & \text{otherwise} \end{cases}
\]

6.2

\( f_{XY}(x, y) = f_X(x) f_Y(y) = \frac{1}{a^2}, \quad 0 < x \leq a, \quad 0 < y \leq a \)

(a)

\[
F_Z(z) = P \left\{ \frac{X}{Y} \leq z \right\} = P\{X \leq zY\}
\]

(i) \( z < 1 \)

\[
F_Z(z) = P\{X \leq zY\}
= \int_0^a \int_0^{yz} \frac{1}{a} \cdot \frac{1}{a} \, dx \, dy = \frac{z}{2}, \quad z \leq 1
\]

(ii) \( z \geq 1 \)

\[
F_Z(z) = P\{X \leq zY\}
= 1 - \int_0^a \int_0^{x/z} \frac{1}{a} \cdot \frac{1}{a} \, dy \, dx
\]

\[
= 1 - \int_0^1 \frac{x}{z} \, dx = 1 - \frac{1}{2z}, \quad z > 1
\]

\[
f_Z(z) = \begin{cases} \frac{1}{2}, & z \leq 1 \\ \frac{1}{2z^2}, & z > 1 \end{cases}
\]
(b) 
\[
F_Z(z) = P(Z \leq z) = P\left\{ \frac{Y}{X+Y} \leq z \right\} \\
= P\left\{ \frac{1}{Y} \geq \frac{1}{z} - 1 \right\} = 1 - P\left( \frac{X}{Y} \leq \frac{1 - z}{z} \right) \\
= \left\{
\begin{array}{ll}
\frac{1}{2} \left( \frac{z}{1 - z} \right), & 0 < z \leq 1/2 \\
1 - \frac{1}{2} \left( \frac{1 - z}{z} \right), & 1/2 < z < 1
\end{array}
\right.
\]

\[
f_Z(z) = \left\{
\begin{array}{ll}
\frac{1}{2(1 - z)^2}, & 0 < z \leq 1/2 \\
\frac{1}{2z^2}, & 1/2 < z < 1
\end{array}
\right.
\]

(c) 
\[
F_Z(z) = P\{Z \leq z\} = P\{|X - Y| \leq z\} \\
= P\{|X - Y| \leq z\} \cap (X \geq Y) + P\{|X - Y| \leq z\} \cap (X < Y) \\
= P\{X - Y \leq z, X \geq Y\} + P\{Y - X \leq z, X < Y\} \\
= \int_0^\infty \int_y^{y+z} f_{XY}(x, y) \, dx \, dy + \int_0^\infty \int_x^{x+z} f_{XY}(x, y) \, dy \, dx \\
= \int_0^\infty \int_y^{y+z} f_{XY}(x, y) \, dx \, dy + \int_0^\infty \int_y^{y+z} f_{XY}(y, x) \, dx \, dy \\
= \int_0^\infty \int_y^{y+z} \{f_{XY}(x, y) + f_{XY}(y, x)\} \, dx \, dy.
\]

In general
\[
f_Z(z) = \int_0^\infty \frac{d}{dz} \int_y^{y+z} f_{XY}(x, y) + f_{XY}(y, x) \, dx \, dy \\
= \int_0^\infty \{f_{XY}(y + z, y) + f_{XY}(y, y + z)\} \, dy.
\]

Here
\[
X \sim U(0, a), \quad Y \sim U(0, a)
\]

\[
F_Z(z) = 1 - \frac{1}{a^2} \cdot 2 \cdot \frac{(a - z)^2}{2} = 1 - \left(1 - \frac{z}{a}\right)^2
\]

and
\[
f_Z(z) = \frac{2}{a} \left(1 - \frac{z}{a}\right) \quad 0 \leq z \leq a.
\]
\[ F_Z(z) = P\{Z \leq z\} = P\{X + Y \leq z\} \]
\[ = \frac{1}{2} - \frac{z^2}{2}, \quad -1 < z < 0, \]

(which represents the area below the line \(X + Y = z\).)

\[ F_Z(z) = P\{Z \leq z\} = P\{X + Y \leq z\} \]
\[ = \frac{1}{2} + \frac{z^2}{2}, \quad 0 \leq z < 1 \]

\[ f_Z(z) = \begin{cases} 
-z, & -1 \leq z < 0 \\
z, & 0 \leq z < 1
\end{cases} \]

6.4

\[ Z = X - Y \]

For \( z < 0 \)

\[ F_Z(z) = P\{Z \leq z\} = \int_0^{(1+z)/2} \int_{-z}^{1-x} f_{XY}(x, y) \, dy \, dx = \int_0^{(1+z)/2} \int_{x-z}^{1-x} 6x \, dy \, dx \]
\[ = \int_0^{(1+z)/2} 6x \left[ y \right]_{x-z}^{1-x} \, dx = \int_0^{(1+z)/2} 6x(1 - x - x + z) \, dx \]
\[ = 6 \left[ \frac{(1+z)^2}{2} - \frac{2z^3}{3} \right]_{0}^{(1+z)/2} = 6 \left\{ \frac{(1+z)^3}{8} - \frac{(1+z)^3}{12} \right\} \]
\[ = \frac{(1+z)^3}{4}, \quad z \leq 0. \]

For \( z > 0 \)

\[ F_Z(z) = P\{Z \leq z\} = 1 - P\{Z > z\} \]
\[ = 1 - \int_0^{(1-z)/2} \int_{-y}^{1-y} f_{XY}(x, y) \, dx \, dy = 1 - \int_0^{(1-z)/2} \int_{z+y}^{1-y} 6x \, dy \, dx \]
\[ = 1 - \int_0^{(1-z)/2} \left[ \frac{6x^2}{2} \right]_{z+y}^{1-y} \, dy = 1 - 3 \int_0^{(1-z)/2} [(1 - y)^2 - (z - y)^2] \, dy \]
\[ = 1 - 3(1 + z) \left[ \frac{(1 - z)^2}{2} - \frac{(1 - z)^2}{4} \right] = 1 - \frac{3}{4} (1 + z)(1 - z)^2 \quad z \leq 0. \]

\[ f_Z(z) = \begin{cases} 
\frac{3}{4} (1 - z)(1 + 3z), & 0 \leq z \leq 1 \\
\frac{3}{4} (1 + z)^2, & -1 < z < 0
\end{cases} \]

6.5 (a) See Example 6-15 for solutions
(b) See Example 6-14 for solutions
(c)

\[ U = X - Y \sim N(0, 2\sigma^2) \]

since linear combinations of jointly Gaussian random variables are Gaussian random variables (see Eq. (6-120) Text.). Here \( Var(U) = Var(X) + Var(Y) = 2\sigma^2 \).
\[ Z = XY \]
\[ F_Z(z) = P(XY \leq z) = 1 - P(XY > z) \]
\[ = 1 - \int_z^1 \int_{z/y}^1 f_{XY}(x, y) \, dx \, dy \]
\[ f_Z(z) = 1 + \int_z^1 \frac{1}{y} f_{XY}(z/y, y) \, dy = 1 + \int_z^1 \left( \frac{2}{y} - \frac{2z}{y^3} \right) \, dy \]
\[ = 1 - 2 \ln z + 2z, \quad 0 \leq z \leq 1 \]

6.7 (a)

\[ Z_1 = X + Y \]
\[ F_{Z_1}(z) = P(X+Y \leq z) = \begin{cases} \int_0^z \int_0^{z-y} f_{XY}(x, y) \, dx \, dy, & 0 < z < 1 \\ 1 - \int_{z-1}^1 \int_{z-y}^1 f_{XY}(x, y) \, dx \, dy, & 1 < z < 2 \end{cases} \]
\[ f_{Z_1}(z) = \begin{cases} \int_0^z f_{XY}(z-y, y) \, dy, & 0 < z < 1 \\ \int_{z-1}^1 f_{XY}(z-y, y) \, dy, & 1 < z < 2 \\ z^2, & 0 < z < 1 \\ z(2-z), & 1 < z < 2 \\ 0, & \text{otherwise} \end{cases} \]

(b)

\[ Z_2 = XY \]
\[ F_{Z_2}(z) = P(XY \leq z) = 1 - \int_z^1 \int_{z/y}^1 f_{XY}(x, y) \, dx \, dy \]
\[ f_{Z_2}(z) = \int_z^1 \frac{1}{y} f_{XY}(z/y, y) \, dy = \int_z^1 \frac{1}{y} \left( \frac{z}{y} + y \right) \, dy \]
\[ = 2(1-z), \quad 0 < z < 1 \]

(c)

\[ Z_3 = \frac{Y}{X} \]
\[ F_{Z_3}(z) = P(Y/X \leq z) = \begin{cases} \int_0^1 \int_0^{xz} f_{XY}(x, y) \, dy \, dx, & 0 < z < 1 \\ 1 - \int_0^1 \int_{y/z}^1 f_{XY}(x, y) \, dx \, dy, & z > 1 \end{cases} \]
\[ f_{Z}(z) = \begin{cases} 
\int_{0}^{1} x f_{XY}(x, z) \, dx, & 0 < z < 1 \\
\int_{0}^{1} \frac{y}{z^2} f_{XY}(y/z, y) \, dy, & z > 1 
\end{cases} \]

\[ = \begin{cases} 
\frac{1+z}{3}, & 0 < z < 1 \\
\frac{1+z}{3z^3}, & z > 1 
\end{cases} \]

(d) \[ Z_4 = Y - X \]

\[ F_{Z_4}(z) = P(Y - X \leq z) = \begin{cases} 
1 - \int_{z}^{1} \int_{0}^{y-z} f_{XY}(x, y) \, dx \, dy, & 0 < z < 1 \\
\int_{z}^{1} \int_{y-z}^{1} f_{XY}(x, y) \, dx \, dy, & -1 < z < 0 
\end{cases} \]

\[ f_{Z_4}(z) = \begin{cases} 
\int_{z}^{1} f_{XY}(y - z, y) \, dy, & 0 < z < 1 \\
\int_{0}^{1} f_{XY}(y - z, y) \, dy, & -1 < z < 0 
\end{cases} \]

\[ = \begin{cases} 
1 - z, & 0 < z < 1 \\
1 + z, & -1 < z < 0 = 1 - |z|, & |z| < 1 
\end{cases} \]

6.8 \[ F_{Z}(z) = P(X + Y \leq z) \]

\[ = \begin{cases} 
\int_{0}^{z/3} \int_{2y}^{z-y} f_{XY}(x, y) \, dx \, dy = \frac{z^2}{6}, & 0 < z < 2 \\
1 - \int_{2z/3}^{2} \int_{z-x}^{x/2} f_{XY}(x, y) \, dy \, dx = 2z - \frac{z^2}{3} - 2, & 2 < z < 3 
\end{cases} \]

Thus \[ f_{Z}(z) = \begin{cases} 
\int_{0}^{z/3} f_{XY}(z - y, y) \, dy, & 0 < z < 2 \\
\int_{2z/3}^{2} f_{XY}(x, z - x) \, dx, & 2 < z < 3 
\end{cases} \]

\[ f_{Z}(z) = \begin{cases} 
\frac{1}{3} z, & 0 < z < 2 \\
2 - \frac{2z}{3}, & 2 < z < 3 \\
0, & \text{otherwise} 
\end{cases} \]
6.9 (a)

\[ Z = \frac{X}{Y}, \quad z \geq 1 \]

\[ F_Z(z) = P(X \leq Yz) = \int_0^1 \int_{x/z}^x f_{XY}(x, y) \, dy \, dx \]

\[ f_Z(z) = \int_0^1 \frac{x}{z^2} f_{XY}(x, x/z) \, dx = \frac{1}{z^2}, \quad z \geq 1 \]

(b)

\[ W = XY \]

\[ F_W(w) = P(W \leq w) = P(XY \leq w) = 1 - P(XY > w) \]

\[ = 1 - \int_{\sqrt{w}}^1 \int_{w/x}^x f_{XY}(x, y) \, dy \, dx \]

Hence

\[ f_W(w) = \int_{\sqrt{w}}^1 \frac{1}{x} f_{XY}(x, w/x) \, dx = \int_{\sqrt{w}}^1 \frac{2}{x^3} \, dx \]

\[ = \ln \left( \frac{1}{w} \right), \quad 0 < w \leq 1 \]

6.10 (a)

\[ Z = X + Y \]

\[ F_Z(z) = \int_0^{z/2} \int_x^{2-x} f_{XY}(x, y) \, dy \, dx = \frac{z^2}{4}, \quad 0 < z < 2 \]

\[ f_Z(z) = \frac{z}{2}, \quad 0 < z < 2 \]

(b)

\[ W = X - Y \]

\[ F_W(w) = \frac{1}{2} (2 + w) \left(1 + \frac{w}{2}\right) = \left(1 + \frac{w}{2}\right)^2 \]

\[ f_W(w) = \begin{cases} 
1 + \frac{w}{2}, & -2 < w < 0 \\
0, & \text{otherwise}
\end{cases} \]
6.11 (a) The characteristic function of \( X + Y \) is given by
\[
\phi_{X+Y}(\omega) = \phi_X(\omega) \phi_Y(\omega) = \frac{1}{(1-j\omega\beta)^\alpha} \cdot \frac{1}{(1-j\omega\beta)^\beta} = \frac{1}{(1-j\omega\beta)^{2\alpha}} \sim \text{Gamma}(2\alpha, \beta)
\]
(b) \( f_{XY}(x, y) = f_X(x) f_Y(y) = \frac{(xy)^{(\alpha-1)}}{\Gamma(\alpha)\beta^\alpha} e^{(x+y)/\beta}, \quad x > 0, \ y > 0 \)
Let
\[
Z = \frac{X}{Y}
\]
Using (Eq. 6-60) we get
\[
f_Z(z) = \int_0^\infty y \left( \frac{y^2 z}{\Gamma(\alpha)\beta^\alpha} \right)^{(\alpha-1)} e^{-(1+z)y/\beta} dy
\]
\[
= \frac{z^{(\alpha-1)}}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty y^{(2\alpha-1)} e^{-(1+z)y/\beta} dy
\]
\[
= \frac{z^{(\alpha-1)}}{\Gamma(\alpha)\beta^\alpha} \frac{\beta^{(2\alpha-1)}}{(1+z)^{2\alpha-1}} \frac{\beta}{(1+z)} \int_0^\infty u^{2\alpha-1} e^{-u} du
\]
\[
= \frac{\Gamma(2\alpha)}{\Gamma(\alpha)\beta^\alpha} z^{\alpha-1}, \quad z > 0
\]
(see also Example 6-27 for the answer).
(c)
\[
W = \frac{X}{X+Y} = \frac{X/Y}{X/Y+1} = \frac{Z}{Z+1}
\]
\[
F_W(w) = P\left(\frac{Z}{Z+1} \leq w\right) = P\left(Z \leq \frac{w}{1-w}\right) = F_Z\left(\frac{w}{1-w}\right)
\]
This gives
\[
f_W(w) = \frac{1}{(1-w)^2} f_Z\left(\frac{w}{1-w}\right)
\]
\[
= \frac{\Gamma(2\alpha)}{\Gamma(\alpha)} w^{\alpha-1} (1-w)^{\alpha-1}
\]
\[
\sim \text{Beta}(\alpha, \alpha)
\]
where we have used results from (b) above.

6.12
\[
X \sim U(0, 1), \quad Y \sim U(0, 1), \quad X, Y \text{ are independent, and}
\]
\[
U = X + Y, \quad V = X - Y \Rightarrow |v| < u < 2.
\]
\( U \) and \( V \) have one pair of solutions given by
\[
x_1 = \frac{u+v}{2}, y_1 = \frac{u-v}{2}.
\]
Also the Jacobian is given by
\[
J = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2
\]
so that
\[
f_{UV}(u, v) = \frac{1}{|J|} f_{XY}(x_1, y_1) = \frac{1}{2}, \quad 0 < |v| < u < 2
\]
\[ f_{XY}(x, y) = \frac{xy}{\sigma^4} e^{-(x^2+y^2)/2\sigma^2}, \quad x, y \geq 0 \]

\[ Z = \frac{X}{Y} \]

\[ F_Z(z) = P(Z \leq z) = P(X/Y \leq z) = \int_0^\infty \int_0^{\infty} f_{XY}(x, y) \, dx \, dy. \]

This gives the density function of \( z \) to be

\[
f_z(z) = \int_0^\infty y f_{XY}(z, y) \, dy = \int_0^\infty \frac{2y^3}{\sigma^4} e^{-(z^2+y^2)/2\sigma^2} \, dy
\]

\[
= \frac{2}{\sigma^4} \int_0^\infty y^3 e^{-y^2(z^2+1)/2\sigma^2} \, dy\quad \text{Let, } t = y^2(z^2+1)/2\sigma^2
\]

\[
= \frac{2}{(z^2+1)^{3/2}} \int_0^\infty t e^{-t} \, dt = \frac{2}{(z^2+1)^{3/2}}, \quad 0 \leq z \leq \infty.
\]

---

6-14

\[ z = x + y \]

\[ f_z(z) = f_x(z) \ast f_y(z) \]

For \( z > 0 \)

\[
2 \int_0^z c^z e^{-cz} = \int_0^z c e^{-c(z-y)} f_y(y) \, dy
\]

\[
c z = \int_0^z e^{cy} f_y(y) \, dy \quad c = e^{cz} f_y(z)
\]

(differentiation). Hence, \( f_y(z) = ce^{-cz} \); and zero for \( z < 0 \).

---

6-15

\[
f_z(z) = \int_{-\infty}^\infty f_x(x) f_y(z-x) \, dx = \int_{z-1}^z f_x(x) \, dx = F_x(z) - F_x(z-1)
\]

because \( f_y(z-x) = 1 \) for \( z-1 < x < z \) and zero otherwise.
All probability masses are on the line \( y = g(x) \).

(a) If \( y = y_1 < g(x) \) then

\[
F(x, y) = P\{x \leq x, \ y \leq y_1 \} = P\{y \leq y_1\} = F_y(y_1).
\]

If \( y = y_2 > g(x) \) then

\[
F(x, y) = P\{x \leq x, \ y \leq y_2 \} = P\{x \leq x\} = F_x(x).
\]

(b) If \( y = y_1 < g(x) \) then

\[
F(x, y) = P\{x \leq x, \ y \leq y_1 \} = 0
\]

If \( y = y_2 > g(x) \) then

\[
F(x, y) = P\{x \leq x, \ y \leq y_2 \} = P\{x \leq x\} - P\{y > y_2\}
\]

\[
= F_x(x) - [1 - F_y(y_2)]
\]
6-17 (a) If \( z = 2x + 3y \) then \( E(z) = 0 \) \[ \sigma_z^2 = 4\sigma_x^2 + 9\sigma_y^2 = 5^2 \]

Hence, \( z \) is \( N(0; 5^2) \)

(b) If \( z = x/y \), then from (6-63) with \( \sigma_1 = \sigma_2 = 2, r = 0 \)

\[ F_z(z) = \frac{1}{2} + \frac{1}{\pi} \arctan z \quad f_z(z) = \frac{1}{\pi(1 + z^2)} \]

6-18

**Diagram:**

The function \( f_{zw}(z,w) = \frac{1}{|x|} f_{xy}(x,y) \) with \( x = w \) and \( y = z/w \)

The function \( f_{zw}(z,w) \) is different from zero in the shaded areas shown. Hence, with \( w^2 - z^2 = s^2 \)

\[ f_z(z) = \frac{1}{\pi^2} \int_{|z|}^{\infty} e^{-w^2/2\alpha^2} \frac{dw}{\sqrt{1-z^2/w^2}} \]

\[ = \frac{1}{\pi^2} \int_{0}^{\infty} e^{-(z^2 + s^2)/2\alpha^2} ds = \frac{1}{\alpha\sqrt{2\pi}} e^{-z^2/2\alpha^2} \]
6-19 (a) \[ z = \frac{x}{y}, \quad w = y \quad \text{J} = 1/y \]

\[ f_z(z) = \int_{-\infty}^{\infty} |w| f_x(zw)f_y(w) \, dw \quad \text{if } z > 0 \]

\[ = \frac{z}{\alpha^2 \beta^2} \int_{0}^{\infty} w^3 e^{-cw^2} \, dw = \frac{z}{2\alpha \beta^2 c^2} \quad c = \frac{z^2}{2\alpha^2} + \frac{1}{2\beta^2} \]

\[ = \frac{2\alpha^2}{\beta^2} \frac{z}{(z^2 + \alpha^2/\beta^2)^2} \quad \text{for } z > 0 \text{ and zero otherwise} \]

(b) \[ F_z(z) = \frac{z}{2\alpha^2} \int_{0}^{z} \frac{dz}{\beta^2 (z^2 + \alpha^2/\beta^2)^2} = \frac{z}{\beta^2} \int_{0}^{\frac{z^2 + \alpha^2/\beta^2}{t^2}} \frac{dt}{\alpha^2/\beta^2} \]

\[ = \frac{z^2}{z^2 + \alpha^2/\beta^2} = P(z \leq z) = P(x < zy) \]
1. The density of $2x$ equals $\frac{1}{2} f_{x}(\frac{x}{2})$. Hence, if $z = 2x + y$, then

$$f_{z}(z) = \int_{0}^{\infty} \frac{a}{\alpha} e^{-ax/2} e^{-\beta(z-x)} dx = \frac{ab}{\alpha - 2\beta}(e^{\beta z} - e^{-\sigma z/2}) U(z)$$

2. The density of $y$ equals $f_{y}(-z)$. Hence, if $z = x - y$, then

$$f_{z}(z) = f_{x}(z) * f_{y}(-z)$$

$$= \begin{cases} \int_{0}^{\infty} e^{-ax} e^{-\beta(x-z)} dx = \frac{ab}{\alpha + \beta} e^{-az} & z > 0 \\ \int_{z}^{\infty} e^{-ax} e^{-\beta(x-z)} dx = \frac{ab}{\alpha + \beta} e^{\beta z} & z < 0 \end{cases}$$

3. $z = x/y$, $\omega = y$, $J = 1/y$

$$f_{z}(z) = ab \int_{0}^{\infty} w e^{-aw} e^{-8w} dw = \frac{ab}{(az + \beta)^{2}} U(z)$$

4. $z = \max(x,y)$, $F_{z}(z) = F_{xy}(z,z) = F_{x}(z)F_{y}(z)$

$$f_{z}(z) = f_{x}(z)F_{y}(z) + f_{y}(z)F_{x}(z)$$

$$= \left[ e^{-az}(1 - e^{-\beta z}) + e^{-\beta z}(1 - e^{-az}) \right] U(z)$$

5. $z = \min(x,y)$, $F_{z}(z) = F_{x}(z) + F_{y}(z) - F_{x}(z)F_{y}(z)$

$$f_{z}(z) = f_{x}(z)[1 - F_{y}(z)] + f_{y}(z)[1 - F_{x}(z)] = (\alpha + \beta)e^{-\alpha + \beta} z U(z)$$
6-22 (a) \[ ay^2 + \beta(x-y)^2 = (a+\beta)(y - \frac{\beta x}{a+\beta})^2 + \frac{a\beta}{a+\beta} x^2 \]

\[ e^{-ax^2} * e^{-bx^2} = \int_{-\infty}^{\infty} e^{-ay^2 - \beta(x-y)^2} dy \]

\[ = e^{-a\beta x^2/(a+\beta)} \int_{-\infty}^{\infty} e^{-(\alpha + \beta)(y - \frac{\beta x}{a+\beta})^2} dy \]

\[ = \sqrt{\frac{\pi}{\alpha + \beta}} e^{-\frac{a\beta x^2}{\alpha + \beta}} \]

(b) \[ \frac{a/\pi}{x^2 + a^2} \times \frac{\beta/\pi}{x^2 + \beta^2} = \frac{a\beta}{\pi} \int_{-\infty}^{\infty} \frac{dy}{(y^2 + a^2)(x^2 + \beta^2)} = \frac{(a+\beta)/\pi}{x^2 + (a+\beta)^2} \]

Characteristic functions lead to a simpler derivation of the above [see (6-192)]
6-23 We introduce the auxiliary variable \( w = y \). The Jacobian of the transformation \( z = nx/ny \), \( w = y \) equals \( n/ny \). Since \( x = mz/n \), \( y = w \) and the RVs \( x \) and \( y \) are independent, (6-1.13) yields
\[
f_{xw}(z, w) = \frac{mw}{n} f_x \left( \frac{m}{n} zw \right) f_y(w) \sim w(zw)^{m/2-1}e^{-msw/2}w^{n/2-1}e^{-w/2}
\]
for \( z > 0 \), \( w > 0 \) and 0 otherwise. Integrating with respect to \( w \), we obtain
\[
f_x(z) \sim z^{m/2-1} \int_0^\infty w^{(m+n)/2-1} \exp \left\{ - \frac{w}{2} \left( 1 + \frac{m}{n} - z \right) \right\} dw
\]
\[
\sim \frac{z^{m/2-1}}{(1+mz/n)(m+n/3)} \int_0^\infty q^{(m+n)/3}e^{-q}dq
\]

6-24

If \( z \leq w \) then
\[
P\{z \leq z, w \leq w\} = P\{z \leq z\} = P\{(x, y) \in D_1\} = F_{xy}(z, z)
\]

If \( z > w \) then
\[
P\{z \leq z, w \leq w\} = P\{(x, y) \in D_2\}
\]
\[
= F_{xy}(z, w) + F_{xy}(w, z) - F_{xy}(w, w)
\]


\[ X \sim \text{Exponential}(\lambda), \quad Y \sim \text{Exponential}(\lambda) \]

\(X\) and \(Y\) are independent so that

\[ f_{XY}(x, y) = f_X(x) f_Y(y) = \frac{1}{\lambda^2} e^{-(x+y)/\lambda} U(x) U(y) \]

\[ Z = X + Y \]

\[ \phi_Z(\omega) = \phi_X(\omega) \phi_Y(\omega) \frac{1}{(1 - j\omega\lambda)^2} \]

\(Z \sim \text{Gamma} (2, \lambda)\)

This gives

\[ f_Z(z) = \frac{z}{\lambda^2} e^{-z/\lambda} U(z) \]

\[ P(Z > 2\lambda) = \int_{2\lambda}^{\infty} \frac{z}{\lambda^2} e^{-z/\lambda} dz = \int_2^{\infty} xe^{-x} dx = 3e^{-2} = 0.406 \]

Let,

\[ W = Y - X \]

Then

\[ P(Y - X > \lambda) = P(W > \lambda) = \int_{\lambda}^{\infty} f_W(w)dw \]

Notice that \(F_W(w)\) is given by (6-55).

For \(w > 0\), this gives

\[ f_W(w) = \int_0^{\infty} \frac{1}{\lambda^2} e^{-(w+2y)/\lambda} dy = \frac{1}{\lambda^2} e^{-w/\lambda} \int_0^{\infty} e^{-2y/\lambda} dy \]

\[ = \frac{1}{2\lambda} e^{-w/\lambda}, \quad w > 0 \]

Hence

\[ P(Y - X > \lambda) = P(W > \lambda) = \int_{\lambda}^{\infty} \frac{1}{2\lambda} e^{-w/\lambda} dw = \frac{1}{2e} \]
6.26 (a)

\[ R = W - Z \]
\[ = \max(X, Y) - \min(X, Y) \]
\[ = \begin{cases} 
X - Y, & X \geq Y \\
Y - X, & X < Y 
\end{cases} \]

\[ F_R(r) = P\{R \leq r\} \]
\[ = P\{R \leq r, X \geq Y\} + P\{R \leq r, X < Y\} \]
\[ = P\{X - Y \leq r, X \geq Y\} + P\{Y - X \leq r, X < Y\} \]
\[ = 1 - 2\left(1 - \frac{r^2}{2}\right) = 1 - (1 - r)^2, \quad 0 \leq r \leq 1 \]

\[ f_R(r) = \begin{cases} 
2(1 - r), & 0 \leq r \leq 1 \\
0, & \text{otherwise} 
\end{cases} \]

(b)

\[ S = W + Z \]
\[ = \max(X, Y) + \min(X, Y) = X + Y \]

Case 1: \(0 < s < 1\)

\[ F_S(s) = P\{S \leq s\} = P\{X + Y \leq s\} = \frac{s^2}{2}, \quad 0 < s < 1 \]

Case 2: \(1 \leq s \leq 2\)

\[ F_S(s) = P\{S \leq s\} = P\{X + Y \leq s\} = 1 - \frac{(2 - s)^2}{2}, \quad 1 \leq s \leq 2 \]

\[ F_S(s) = \begin{cases} 
s, & 0 \leq s \leq 1 \\
(2 - s), & 1 \leq s \leq 2 \\
0, & \text{otherwise} 
\end{cases} \]
6.27 (a) $X, Y$ are independent, identically distributed exponential random variables.

$$Z = \frac{Y}{\max(X, Y)} = \begin{cases} \frac{Y}{X}, & X \geq Y \\ 1, & X < Y \end{cases} \Rightarrow 0 < z \leq 1.$$  

$0 < z < 1$

$$F_Z(z) = P(Z \leq z) = P\left\{ \frac{Y}{X} \leq z, X > Y \right\}$$

$$= P\{Y \leq Xz, X > Y\} = \int_0^\infty \int_0^{xz} f_{XY}(x, y) \, dy \, dx$$

$$f_Z(z) = \int_0^\infty x f_{XY}(x, xz) \, dx = \int_0^\infty \frac{x}{\lambda^2} e^{-(1+z)x/\lambda} \, dx = \frac{1}{(1+z)^2}, \quad 0 < z < 1.$$  

Also

$$P(Z = 1) = P(X < Y) = \int_0^\infty \int_0^y \frac{1}{\lambda^2} e^{-(x+y)/\lambda} \, dx \, dy = \frac{1}{2}$$

(b)

$$W = \frac{X}{\min(X, 2Y)} = \begin{cases} \frac{X}{2Y}, & X \geq 2Y \\ 1, & X < 2Y \end{cases} \Rightarrow 1 \leq w < \infty$$

$$F_W(w) = P(X \leq 2Yw, X > 2Y) = \int_0^\infty \int_{2y}^{2wy} f_{XY}(x, y) \, dx \, dy$$

This gives

$$f_W(w) = \int_0^\infty 2y f_{XY}(2wy, y) \, dy = \int_0^\infty \frac{2y}{\lambda^2} e^{-(1+2w)y/\lambda} \, dy$$

$$= \frac{2}{(1+2w)^2}, \quad w > 1$$

Also

$$P(W = 1) = P(X < 2Y) = \int_0^\infty \int_0^{2y} \frac{1}{\lambda^2} e^{-(x+y)/\lambda} \, dx \, dy = \frac{2}{3}$$

Note that the p.d.f. of $Z$ as well as $W$ has an impulse at $z = 1$ and $w = 1$ respectively.
6.28 \( X, Y \) are independent identically distributed exponential random variables.

\[
Z = \frac{X}{X + Y}
\]

\[
F_Z(z) = P\left( \frac{X}{X + Y} \leq z \right) = P\left( \frac{Y}{1 - z} \leq 1 - \frac{z}{1 - z} \right)
\]

\[
= P\left( X \leq \frac{zY}{1 - z} \right) = \int_0^\infty \int_{0}^{(zy)/(1-z)} f_{XY}(x, y) \, dx \, dy
\]

\[
f_Z(z) = \int_0^\infty \frac{y}{(1-z)^2} f_{XY}(zy/(1-z), y) \, dy
\]

\[
= \frac{1}{(1-z)^2} \int_0^\infty \frac{y}{\lambda^2} e^{-(z/(1-z)+1)(y/\lambda)} \, dy
\]

\[
= \frac{1}{(1-z)^2} \int_0^\infty \frac{y}{\lambda^2} e^{-[y/(1-z)\lambda]} \, dy
\]

\[
= \int_0^\infty u e^{-u} \, du = 1, \quad 0 < z < 1
\]

\[
\Rightarrow \frac{X}{X + Y} \sim U(0, 1)
\]

6.29 Let

\[
f_X(x) = \frac{1}{\lambda} e^{-x/\lambda} U(x), \quad f_Y(y) = \frac{1}{\lambda} e^{-y/\lambda} U(y).
\]

\[
Z = \min(X, Y)
\]

\[
W = \max(X, Y) - \min(X, Y)
\]

\[
Z = \begin{cases} Y, & X \geq Y \\ X, & X < Y \end{cases}
\]

\[
W = \begin{cases} X - Y, & X \geq Y \\ Y - X, & X < Y \end{cases}
\]

\( Z = \min(X, Y) \). See Example 6-18, Eq.(6-82) for solution. From there (replace \( \lambda \) by \( 1/\lambda \) in (6-82))

\[
f_Z(z) = \frac{2}{\lambda} e^{-2z/\lambda} U(z).
\]

\[
F_W(w) = P(X - Y \leq w, X \geq Y) + P(Y - X \leq w, X < Y)
\]

\[
= \int_0^\infty \int_{y}^{y+w} f_{XY}(x, y) \, dx \, dy
\]

\[
+ \int_0^\infty \int_{x}^{x+w} f_{XY}(x, y) \, dy \, dx, \quad w > 0
\]

This gives

\[
F_W(w) = \int_0^\infty f_{XY}(y + w, y) \, dy + \int_0^\infty f_{XY}(x, x + w) \, dx
\]

\[
= 2 \int_0^\infty \frac{1}{\lambda^2} e^{(2y+w)/\lambda} \, dy
\]

\[
= \frac{2}{\lambda^2} e^{-w/\lambda} \left[ \frac{e^{-2y/\lambda}}{-2/\lambda} \right]_0^\infty = \frac{1}{\lambda} e^{-w/\lambda}, \quad w > 0
\]
Also
\[ F_{ZW}(z, w) = P\{Z \leq z, W \leq w\} \]
\[ = P\{Y \leq z, X - Y \leq w, X \geq Y\} \]
\[ + P\{X \leq z, Y - X \leq w, X < Y\} \]
\[ = \int_0^z \int_y^{y+w} f_{XY}(x, y) \, dx \, dy + \int_0^z \int_x^{x+w} f_{XY}(x, y) \, dy \, dx \]

Repeated use of (6-39)-(6-40) gives
\[ f_{ZW}(z, w) = f_{XY}(z + w, z) + f_{XY}(z, z + w) \]
\[ = \frac{2}{\lambda^2} e^{-(z+w)/\lambda} = \frac{2}{\lambda} e^{-2z/\lambda} \frac{1}{\lambda} e^{-w/\lambda} \]
\[ = f_Z(z) f_W(w) \]

Thus Z and W are independent exponential random variables.

6.30 (a) Let
\[ U = X + Y, \quad 0 < u < 2\beta. \]

The probability density function of U can be computed as in (6-48)-(6-50). Using Fig. 6-11, for 0 < u ≤ β, we have
\[ F_U(u) = \int_0^u \int_0^{u-x} f_{XY}(x, y) \, dy \, dx \]

which gives
\[ f_U(u) = \int_0^u f_{XY}(x, u-x) \, dx = \alpha^2 \beta^{-2\alpha} \int_0^u x^{\alpha-1} (u-x)^{\alpha-1} \, dx \]
\[ = \alpha^2 \beta^{-2\alpha} u^{2\alpha-1} \int_0^1 y^{\alpha-1} (1-y)^{\alpha-1} \, dy \]
\[ = B(\alpha, \alpha) \alpha^2 \beta^{-2\alpha} u^{2\alpha-1}, \quad 0 < u \leq \beta \]

where we have substituted y = ux and made use of the beta function defined in (4-49)-(4-51). Similarly for β < u ≤ 2β, we get (see (6-49))
\[ F_U(u) = 1 - \int_{u-\beta}^\beta \int_0^{u-x} f_{XY}(x, y) \, dy \, dx \]

and hence
\[ f_U(u) = \int_{u-\beta}^{\beta} f_{XY}(x, u-x) \, dx = \alpha^2 \beta^{-2\alpha} \int_{u-\beta}^\beta x^{\alpha-1} (u-x)^{\alpha-1} \, dx \]
\[ = \alpha^2 \beta^{-2\alpha} u^{2\alpha-1} \int_{1-\beta/u}^{\beta/u} y^{\alpha-1} (1-y)^{\alpha-1} \, dy, \quad \beta < u \leq 2\beta \]

(b)
\[ Z = \min(X, Y), \quad W = \max(X, Y) \]

We can proceed as in Example 6-21 to complete this problem. From (6-92) and (6-93), we get
\[ F_{ZW}(z, w) = \begin{cases} 
F_{XY}(z, w) + F_{XY}(w, z) - F_{XY}(z, z), & w \geq z \\
F_{XY}(w, w), & w < z 
\end{cases} \]
which gives
\[ f_{ZW}(z, w) = f_X(z)f_Y(w) + f_X(w)f_Y(z), \quad 0 < z \leq w < \beta \]
\[ f_{ZW}(z, w) = \begin{cases} 
2\alpha^2\beta^{-2a}z^{a-1}w^{a-1}, & 0 < z \leq w < \beta \\
0, & \text{otherwise}
\end{cases} \]

check:
\[
\int_0^\beta \int_0^w f_{ZW}(z, w) dz dw = 2\alpha^2\beta^{-2a} \int_0^\beta w^{a-1} \left( \frac{z^\alpha}{\alpha} \right)^w dw
\]
\[
= 2\alpha\beta^{-2a} \int_0^\beta w^{2a-1} dw = 1
\]

Note: \( Z \) and \( W \) are not independent random variables, since
\[ f_Z(z) = 2\alpha\beta^{-2a} z^{a-1} (\beta^\alpha - z^\alpha), \quad 0 < z < \beta \]
and
\[ f_W(w) = 2\alpha\beta^{-2a} w^{2a-1}, \quad 0 < w < \beta \]

(c) Let
\[ V = \frac{Z}{W} = \frac{\min(X, Y)}{\max(X, Y)} = \begin{cases} 
Y, & X \geq Y \\
X, & X < Y
\end{cases} \]

and
\[ W = \max(X, Y) = \begin{cases} 
X, & X \geq Y \\
Y, & X < Y
\end{cases} \]

For \( 0 < v < 1, \ 0 < w < \beta \)
\[ F_{VW}(v, w) = P(V \leq v, W \leq w) \]
\[ = P\{V \leq v, W \leq w, (X \geq Y) \cup (X < Y)\} \]
\[ = P\{Y \leq Xv, X \leq w, X \geq Y\} + P\{X < Yv, Y \leq w, X < Y\} \]
\[ = \int_0^v \int_0^{xv} f_{XY}(x, y) dy \ dx + \int_v^w \int_0^{yw} f_{XY}(x, y) dx \ dy \]
Hence
\[
f_{VW}(v, w) = \frac{\partial^2 F_{VW}(v, w)}{\partial v \partial w}
= \frac{\partial}{\partial v} \left\{ \int_0^w f_{XY}(w, y) \, dy + \int_0^w f_{XY}(x, w) \, dx \right\}
= w \{f_{XY}(w, v) + f_{XY}(v, w)\}
= 2\alpha^2 \beta^{-2\alpha} w^{2\alpha-1} v^{\alpha-1}, \quad 0 < v < 1, \ 0 < w < \beta
\]

Hence
\[
f_V(v) = \int_0^v f_{VW}(v, w) \, dw = \alpha v^{\alpha-1}, \quad 0 < v < 1
\]
\[
f_W(w) = \int_0^1 f_{VW}(v, w) \, dv = 2\alpha \beta^{-2\alpha} w^{2\alpha-1}, \quad 0 < w < \beta
\]
and
\[
f_{VW}(v, w) = f_V(v) f_W(w).
\]
Thus $V$ and $W$ are independent random variables.

6.31 (a) Solved in Examples 6-27 and 6-12.
(b) Solved in Example 6-27.
(c)
\[
Z = X + Y, \quad W = \frac{X}{X + Y}
\]
\[
x_1 = zw, \quad y_1 = z - x_1 = z(1 - w)
\]
\[
J = \left| \begin{array}{cc}
1 & 1 \\
-\frac{y}{(x + y)^2} & -\frac{x}{(x + y)^2}
\end{array} \right| = \frac{1}{x + y} = \frac{1}{z}
\]
\[
f_{ZW}(z, w) = \frac{z}{\alpha^{m+n} \Gamma(m) \Gamma(n)} (zw)^{m-1} (z(1 - w))^{n-1}
= \left(\frac{z^{m+n-1}}{\alpha^{m+n} \Gamma(\alpha + \beta)} e^{-z/\alpha}\right) \left(\frac{\Gamma(m + n)}{\Gamma(m) \Gamma(n)} \right) w^{m-1} (1 - w)^{n-1}
= f_Z(z) f_W(w)
\]
Thus $Z$ and $W$ are independent random variables.
6.32 (a)

\[ Z = \frac{X}{|Y|}, \quad W = \frac{|X|}{|Y|} = |Z| \]

\[ F_Z(z) = P(Z \leq z) = P(X \leq |Y|z) = \int_{-\infty}^{\infty} \int_{0}^{\frac{|y|z}{\sigma^2}} f_{XY}(x, y) \, dx \, dy \]

\[ = 2 \int_{0}^{\infty} y f_{XY}(\frac{|y|z}{\sigma^2}, y) \, dy = \frac{2}{2\pi\sigma^2} \int_{0}^{\infty} ye^{-(z^2+1)y^2/2\sigma^2} \, dy \]

\[ = \frac{1}{\pi(1+z^2)}, \quad -\infty < z < \infty \]

Thus \( Z \) is a Cauchy random variable. Interestingly, the random variable \( X/Y \) is also a Cauchy random variable (see Example 6-11).

\[ W = |Z| \]

so that

\[ F_W(w) = P(W \leq w) = P(|Z| \leq w) \]

\[ = P(-w < Z < w) = F_Z(w) - F_Z(-w) \]

and hence

\[ f_W(w) = f_Z(w) + f_Z(-w) = \frac{2}{\pi(1+w^2)}, \quad w > 0. \]

(b)

\[ U = X + Y \sim N(0, 2) \]

\[ V = X^2 + Y^2 \sim \text{Exponential} (2) \]

(see Example 6-14). Here \( U, V \) are not independent, since

\[ J(x, y) = \begin{vmatrix} 1 & 1 \\ 2x & 2y \end{vmatrix} = -2(x - y) = 2\sqrt{2v - u^2} \]

and

\[ f_{UV}(u, v) = \frac{1}{2\sqrt{2v - u^2}} \frac{1}{2\pi\sigma^2} e^{-v/2\sigma^2} \]

\[ \neq f_U(u) f_V(v), \quad -\infty < u < \infty, \quad v > 0. \]

6.33

\[ Z = X + Y, \quad W = X - Y \]

are jointly normal random variables. Hence if they are uncorrelated, then they are also independent.

\[ Cov(Z, W) = E[(Z - \mu_Z)(W - \mu_W)] \]

\[ = E \left[ \{(X - \mu_X) + (Y - \mu_Y)\} \{(X - \mu_X) - (Y - \mu_Y)\} \right] \]

\[ = \text{Var}(X) - \text{Var}(Y) = \sigma_X^2 - \sigma_Y^2. \]

The random variables \( Z \) and \( W \) are uncorrelated implies that \( Cov(Z, W) = 0 \). Hence \( \sigma_X^2 = \sigma_Y^2 \) is the necessary and sufficient condition for the independence of \( X + Y \) and \( X - Y \).
6.34 (a)-(b) Let

\[ R = \sqrt{X^2 + Y^2}, \quad \theta = \tan^{-1} \left( \frac{Y}{X} \right) \]

Form Example 6-22, \( R \) and \( \theta \) are independent random variables with joint p.d.f. as in (6-128). (see (6-131)). In term of \( R \) and \( \theta \), we have \( X = R \cos \theta, Y = R \sin \theta \) and hence we obtain

\[ U = \frac{X^2 - Y^2}{\sqrt{X^2 + Y^2}} = R \cos 2\theta \]

\[ V = \frac{2XY}{\sqrt{X^2 + Y^2}} = R \sin 2\theta \]

This gives

\[ J = \begin{vmatrix} \cos 2\theta & -2r \sin 2\theta \\ \sin 2\theta & 2r \cos 2\theta \end{vmatrix} = 2r = 2\sqrt{u^2 + v^2} \]

\[ r = \sqrt{u^2 + v^2}, \quad \theta_1 = \frac{1}{2} \tan^{-1} \left( \frac{v}{u} \right), \quad 2\theta_2 = \pi + 2\theta_1. \]

There are two sets of solutions \((r, \theta_1)\) and \((r, \theta_2)\). Substituting into (6-128) we get

\[ f_{UV}(u, v) = \frac{1}{|J|} \left\{ f_{r, \theta}(r, \theta_1) + f_{r, \theta}(r, \theta_2) \right\} = \frac{2}{|J|} f_{r, \theta}(r, \theta_1) \]

\[ = \frac{2}{2\sqrt{u^2 + v^2}} \frac{\sqrt{u^2 + v^2}}{2\pi \sigma^2} e^{-(u^2+v^2)/2\sigma^2} \]

\[ = \frac{1}{2\pi \sigma^2} e^{-(u^2+v^2)/2\sigma^2} = \frac{1}{\sqrt{2\pi \sigma^2}} e^{-u^2/2\sigma^2} \cdot \frac{1}{\sqrt{2\pi \sigma^2}} e^{-v^2/2\sigma^2} = f_U(u) f_V(v) \]

Thus \( U \) and \( V \) are independent normal random variables. Hence it follows that \( U = \frac{X^2 - Y^2}{\sqrt{X^2 + Y^2}} \) and \( V/2 = \frac{XY}{\sqrt{X^2 + Y^2}} \) are independent random variables.

(c)

\[ Z = \frac{(X - Y)^2 - 2Y^2}{\sqrt{X^2 + Y^2}} = \frac{(X^2 - Y^2) - 2XY}{\sqrt{X^2 + Y^2}} \]

\[ = \frac{X^2 - Y^2}{\sqrt{X^2 + Y^2}} - \frac{2XY}{\sqrt{X^2 + Y^2}} \]

\[ = U - V \sim N(0, 2\sigma^2). \]
6.35 (a) \( Z \sim F(m, n) \) is given by (6-157) Let

\[
Y = \frac{1}{Z}
\]

Then

\[
F_Y(y) = \left| \frac{dy}{dz} \right| f_Z(1/y)
\]

\[
= \frac{1}{y^2} \left( \frac{m/n}{\beta(m/2, n/2)} \right)^{m/2} \frac{1}{y^{m/2-1}} \left( 1 + m/ny \right)^{-(m+n)/2}
\]

\[
= \frac{(n/m)^{m/2}}{\beta(n/2, m/2)} y^{n/2-1} \left( 1 + \frac{n}{my} \right)^{-(m+n)/2}
\]

\[
\sim F(n, m).
\]

(b)

\[
W = \frac{Zm}{Zm + n}
\]

\[
F_W(w) = P(W \leq w) = P \left( \frac{Zm}{Zm + n} \leq w \right)
\]

\[
= P \left( Z \leq \frac{nw}{m(1 - w)} \right) = F_Z \left( \frac{nw}{m(1 - w)} \right)
\]

which gives

\[
f_W(w) = \frac{n}{m(1 - w)^2} f_Z \left( \frac{nw}{m(1 - w)} \right)
\]

\[
= \frac{n}{m(1 - w)^2} \frac{(m/n)^{m/2}}{\beta(m/2, n/2)} \left( \frac{nw}{m(1 - w)} \right)^{m/2-1} \left( 1 + \frac{w}{1 - w} \right)^{-(m+n)/2}
\]

\[
= \frac{1}{\beta(m/2, n/2)} w^{m/2-1} (1 - w)^{n/2-1}, \quad 0 < w < 1.
\]

Thus \( W \) has Beta distribution.

6.36

\[ Z = X + Y > 0, \quad W = X - Y > 0 \]

\[ x_1 = \frac{z + w}{2}, \quad y_1 = \frac{z - w}{2} \]

is the only solution. Moreover

\[
J = \left| \begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array} \right| = -2
\]

so that

\[
f_{ZW}(z, w) = \frac{1}{|J|} f_{XY}(x_1, y_1) = \frac{1}{2} e^{-(z+w)/2}, \quad 0 < w < z < \infty
\]

\[
F_Z(z) = \int_0^z f_{ZW}(z, w) \, dw = \frac{1}{2} e^{-z/2} \left. \frac{e^{-w/2}}{(-1/2)} \right|_0^z
\]

\[
= \frac{1}{2} e^{-z/2} \left. \frac{e^{-w/2}}{(-1/2)} \right|_0^z = e^{-z/2}(1 - e^{-z/2}), \quad z > 0
\]

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\[ Z = X + Y > 0, \quad W = \frac{Y}{X} > 1 \]

\[ y = xw, \quad x(1 + w) = z, \quad x_1 = \frac{z}{1 + w}, \quad y_1 = \frac{zw}{1 + w} \]

is the only solution. Also

\[
J = \begin{vmatrix}
1 & 1 \\
\frac{y}{x^2} & \frac{1}{x}
\end{vmatrix} = \frac{x + y}{x^2} = \frac{(1 + w)^2}{z}
\]

This gives

\[
f_{ZW}(z, w) = \frac{1}{|J|} f_{XY}(x_1, y_1) = \frac{z}{(1 + w)^2} 2e^{-z}, \quad z > 0, \; w > 1
\]

\[
= ze^{-z} \frac{2}{(1 + w)^2} = f_Z(z) f_W(w)
\]

since

\[
f_Z(z) = \int_1^\infty f_{ZW}(z, w) \, dw
\]

\[
= 2ze^{-z} \int_1^\infty \frac{1}{(1 + w)^2} \, dw = z e^{-z}, \quad z > 0
\]

and

\[
f_W(w) = \int_0^\infty f_{ZW}(z, w) \, dz
\]

\[
= \frac{2}{(1 + w)^2} \int_0^\infty z e^{-z} \, dz = \frac{2}{(1 + w)^2}, \quad w > 1.
\]

Thus \( Z \) and \( W \) are independent random variables.
6-38 \[ z = xy \quad y = \cos(\omega t + \theta) \]
\[ w = y \quad J = |y| \quad f_y(y) = \begin{cases} \frac{1}{\pi \sqrt{1 - y^2}} & |y| < 1 \\ 0 & |y| > 1 \end{cases} \]

The RVs \( x \) and \( y \) are independent. Hence,

\[
f_{zw}(z,w) = \frac{1}{|w|} f_x(z) f_y(w) \]

\[
f_z(z) = \frac{1}{\pi} \int_{-1}^{1} \frac{f_x(z/w)}{|w| \sqrt{1 - w^2}} \, dw = \frac{1}{\pi} \int_{|x| > z} \frac{f_x(x)}{\sqrt{x^2 - z^2}} \, dx \]

6-39 \[ z = x + s \quad s = a \cos y \]

\[
f_z(z) = f_x(z) * f_s(s) \quad f_s(s) = \begin{cases} \frac{1}{\pi \sqrt{\alpha^2 - s^2}} & |s| < \alpha \\ 0 & |s| > \alpha \end{cases} \]

\[
f_z(z) = \frac{1}{\pi \sigma \sqrt{2\pi}} \int_{-\alpha}^{\alpha} \frac{e^{-(z-s)^2/2\sigma^2}}{\sqrt{\alpha^2 - s^2}} \, ds = \frac{1}{\pi \sigma \sqrt{2\pi}} \int_{-\pi}^{\pi} e^{-(z - a \cos y)^2/2\sigma^2} \, dy \]

6-40

**Point masses**

\[ P(x = k, y = n - k) = a_k b_{n-k} \]

\[ \{z = n\} = \sum_{k=0}^{n} \{x = k, y = n - k\} \]

\[ P(z = n) = \sum_{k=0}^{n} P(x = k, y = n - k) \]
Line masses

\[ p(x = x_n, \frac{w}{x_n} < \frac{y}{x_n} < \frac{w + \Delta w}{x_n}) = p_n f_y \left( \frac{w}{x_n} \right) \frac{\Delta w}{x_n} \]

\[ \{ \frac{z}{x_n} < \frac{y}{x_n} < \frac{z + \Delta z}{x_n} \} = \sum_n \{ x = x_n, \frac{z - x_n}{x_n} < \frac{y}{x_n} \leq \frac{z + x_n}{x_n} \} \]

\[ f_z(z) \Delta z = \sum_n p_n f_y \left( z - x_n \right) \Delta z \]
6.42 \( X, Y \) are independent geometric random variables. Thus

\[
P\{X = k, Y = m\} = P\{X = k\} \cdot P\{Y = m\} = (pq^k)(pq^m) = p^2q^{k+m}, \quad k, m = 0, 1, 2, \ldots
\]

(a) Let

\[
Z = X + Y
\]

\[
P\{Z = n\} = P\{X + Y = n\} = \sum_k P\{X = k, Y = n - k\}
\]

\[
= \sum_{k=0}^{n} P\{X = k, Y = n - k\}
\]

\[
= \sum_{k=0}^{n} P\{X = k\} \cdot P\{Y = n - k\}
\]

\[
= \sum_{k=0}^{n} pq^k \cdot pq^{n-k} = \sum_{k=0}^{n} p^2 q^n
\]

\[
= (n + 1) p^2 q^n, \quad n = 0, 1, 2, \ldots
\]

(b) Let

\[
W = X - Y
\]

Case 1: \( W \geq 0 \Rightarrow X \geq Y \). Thus for \( m \geq 0 \)

\[
P\{W = m\} = P\{X - Y = m\} = \sum_{k=0}^{\infty} P\{X = m + k, Y = k\}
\]

\[
= \sum_{k=0}^{\infty} P\{X = m + k, Y = k\}
\]

\[
= \sum_{k=0}^{\infty} P\{X = m + k\} \cdot P\{Y = k\}
\]

\[
= \sum_{k=0}^{\infty} (pq^{m+k})(pq^k) = p^2q^m \sum_{k=0}^{\infty} q^{2k}
\]

\[
= p^2q^m (1 + q^2 + q^4 + \cdots) = \frac{p^2q^m}{1 - q^2}
\]

\[
= \frac{pq^m}{1 + q}, \quad m = 0, 1, 2, \ldots \tag{1}
\]

Case 2: \( W < 0 \Rightarrow X < Y \). Thus for \( m < 0 \)

\[
P\{W = m\} = P\{X - Y = m\} = \sum_k P\{X = k, Y = k - m\}
\]

\[
= \sum_{k=0}^{\infty} P\{X = k, Y = k - m\}
\]

\[
= \sum_{k=0}^{\infty} P\{X = k\} \cdot P\{Y = k - m\}
\]

\[
= \sum_{k=0}^{\infty} (pq^k)(pq^{k-m}) = p^2q^{-m} \sum_{k=0}^{\infty} q^{2k}
\]

\[
= \frac{p^2q^{-m}}{1 - q^2} = \frac{pq^{-m}}{1 + q}, \quad m = -1, -2, \ldots \tag{2}
\]

Thus combining (1) and (2) we can write

\[
P\{W = m\} = \frac{pq^{|m|}}{1 + q}, \quad m = 0, \pm 1, \pm 2, \ldots
\]
6.43 We have $X$ and $Y$ are independent and $P(X = k) = P(Y = k) = p_k$. Also

$$P(X = k|X + Y = k) = \frac{P(X = k, Y = 0)}{P(X + Y = k)} = \frac{p_k p_0}{\sum_{i=0}^k p_i p_{k-i}} = \frac{1}{k + 1}. \quad (1)$$

Also

$$P(X = k - 1|X + Y = k) = \frac{P(X = k - 1, Y = 1)}{P(X + Y = k)} \cdot \frac{p_{k-1} p_1}{\sum_{i=0}^k p_i p_{k-i}} = \frac{1}{k + 1}. \quad (2)$$

From (1) and (2),

$$\frac{p_k}{p_{k-1}} = \frac{p_1}{p_0} \Rightarrow p_k = \lambda p_{k-1} = \lambda^k p_0$$

where $\lambda \triangleq p_1/p_0$. Since $\sum_{k=0}^{\infty} p_k = 1$, we must have $\lambda < 1$, and this gives

$$\sum_{k=0}^{\infty} p_k = \frac{p_0}{1 - \lambda} = 1 \Rightarrow p_0 = 1 - \lambda.$$ 

Thus

$$p_k = p_0 \lambda^k = (1 - \lambda) \lambda^k, \quad k = 0, 1, 2, \cdots, \ 0 < \lambda < 1$$

represents a geometric distribution. Thus $X$ and $Y$ are geometric random variables.

6.44 The moment generating functions of $X$ and $Y$ are given by (see (5-117))

$$\Gamma_X(z) = (pz + q)^n, \quad \Gamma_Y(z) = (pz + q)^n$$

Also

$$\Gamma_{X+Y}(z) = E[z^{X+Y}] = \Gamma_X(z)\Gamma_Y(z) = (pz + q)^{2n} \sim \text{Binomial} (2n, p)$$
6.45 (a) Let
\[ Z = \min(X, Y), \quad W = X - Y \]
\[
P\{Z = k, W = m\} = P\{\min(X, Y) = k, X - Y = m\}
\]
\[
= P\{(\min(X, Y) = k, X - Y = m) \cap (X \geq Y \cup X < Y)\}
\]
\[
= P\{Y = k, X - Y = m, X \geq Y\} + P\{X = k, X - Y = m, X < Y\}
\]
\[
= P\{X = m + k, Y = k, X \geq Y\} + P\{X = k, Y = k - m, X < Y\}
\]

Note that \( k \geq 0 \), and \( m \) takes both positive, zero and negative values. Hence

\[
P\{Z = k, W = m\} = \begin{cases} 
P\{X = k + m, Y = k, X \geq Y\}, & k \geq 0, \ m \geq 0 \\
P\{X = k, Y = k - m, X < Y\}, & k \geq 0, \ m < 0 
\end{cases}
\]

\[
P\{Z = k, W = m\} = \begin{cases} 
pq^{k+m}pq^{k}, & k \geq 0, \ m \geq 0 \\
pq^{k-m}pq^{k}, & k \geq 0, \ m < 0 
\end{cases}
\]

\[
P\{Z = k, W = m\} = p^2q^{2k+|m|}, \quad k = 0, 1, 2, \cdots, \ m = 0, \pm 1, \pm 2, \cdots
\]

Also

\[
P\{Z = k\} = \sum_{m=-\infty}^{\infty} P\{Z = k, W = m\}
\]
\[
= p^2q^{2k} \sum_{m=-\infty}^{\infty} q^{|m|} = p^2q^{2k} \left( 1 + 2 \sum_{m=1}^{\infty} q^m \right)
\]
\[
= p^2q^{2k} \left( 1 + \frac{2q}{p} \right) = pq(1+q)^{2k}, \quad k = 0, 1, 2, \cdots
\]

and

\[
P\{W = m\} = \sum_{k=0}^{\infty} P\{Z = k, W = m\}
\]
\[
= p^2q^{|m|} \sum_{k=0}^{\infty} q^{2k}
\]
\[
= \frac{p}{1+q^{2|m|}}, \quad m = 0, \pm 1, \pm 2, \cdots
\]
Note that
\[ P\{Z = k, W = m\} = P\{Z = k\} P\{W = m\} \]
and hence \( Z \) and \( W \) are independent random variables.

(b) Let
\[ Z = \min(X, Y), \quad W = \max(X, Y) - \min(X, Y) \]
Proceeding as in (a), we obtain
\[
P\{Z = k, W = m\} \\
= P(Y = k, X - Y = m, X \geq Y) + P(X = k, Y - X = m, X < Y) \\
= P(X = k + m, Y = k, X \geq Y) + P(X = k, Y = k + m, X < Y) \\
= \begin{cases} 
pq^{k+m} + pq^k pq^{k+m}, & k = 0, 1, 2, \ldots, m = 1, 2, \ldots \\
pq^{k+m}, & k = 0, 1, 2, \ldots, m = 0 \\
2p^2 q^{2k+m}, & k = 0, 1, 2, \ldots, m = 1, 2, \ldots \\
p^2 q^{2k}, & k = 0, 1, 2, \ldots, m = 0 \end{cases}
\]
This gives
\[
P\{Z = k\} = \sum_{m=0}^{\infty} P\{Z = k, W = m\} \\
= p^2 q^{2k} \left( 1 + 2\sum_{m=1}^{\infty} q^m \right) = p^2 q^{2k} \left( 1 + \frac{2q}{p} \right) \\
= p(1 + q)q^{2k}, \quad k = 0, 1, 2, \ldots
\]
Also
\[
P\{W = m\} = \sum_{k=0}^{\infty} P\{Z = k, W = m\} \\
= \begin{cases} 
p \frac{p}{1+q}, & m = 0 \\
2p \frac{q^m}{1+q}, & m = 1, 2, \ldots \end{cases}
\]
Notice that
\[ P\{Z = k, W = m\} = P\{Z = k\} P\{W = m\} \]
and hence \( Z \) and \( W \) are also independent random variables in this case also.
6.46 The moment generating function of \( X \) and \( Y \) are given by (see (5-119))

\[
\Gamma_X(z) = e^{\lambda_1(z-1)}, \quad \Gamma_Y(z) = e^{\lambda_2(z-1)}
\]

Also

\[
\Gamma_{X+Y}(z) = \Gamma_X(z)\Gamma_Y(z) = e^{(\lambda_1+\lambda_2)(z-1)}
\]

so that

\[
Z \sim P(\lambda_1 + \lambda_2)
\]

Thus

\[
P(X + Y = k) = e^{-(\lambda_1+\lambda_2)} \frac{(\lambda_1 + \lambda_2)^k}{k!}
\]

and

\[
P(X = k|X + Y = n) = \frac{P(X = k, X + Y = n)}{P(X + Y = n)} = \frac{P(X = k)P(Y = n-k)}{P(X + Y = n)}
\]

\[
= \frac{e^{-\lambda_1}(\lambda_1^k/k!)}{e^{-(\lambda_1+\lambda_2)}(\lambda_1 + \lambda_2)^n/n!} \frac{e^{-\lambda_2}(\lambda_2^{n-k}/(n-k)!)}{e^{-(\lambda_1+\lambda_2)}(\lambda_1 + \lambda_2)^n/n!}
\]

\[
= \binom{n}{k} \left( \frac{\lambda_1}{\lambda_1 + \lambda_2} \right)^k \left( \frac{\lambda_2}{\lambda_1 + \lambda_2} \right)^{n-k}, \quad k = 0, 1, 2, \ldots n
\]

\[
\sim \text{Binomial}(n, p), \text{ where } p = \frac{\lambda_1}{\lambda_1 + \lambda_2}.
\]

See also (6-222). From there the converse is also true (proceed as in Example 6-43).
\[
\begin{align*}
C &= \begin{bmatrix}
\sigma_1^2 & r\sigma_1\sigma_2 \\
 r\sigma_1\sigma_2 & \sigma_2^2 \\
\end{bmatrix} \\
\Delta &= \frac{\sigma_1^2 \sigma_2^2 (1 - r^2)}{
\begin{bmatrix}
\frac{1}{(1-r^2)\sigma_1^2} & \frac{r}{(1-r^2)\sigma_1\sigma_2} \\
\frac{r}{(1-r^2)\sigma_1\sigma_2} & \frac{1}{(1-r^2)\sigma_2^2} \\
\end{bmatrix}
\end{align*}
\]

\[
xC^{-1}x^t = \frac{1}{(1-r^2)} \left( \frac{x_1^2}{\sigma_1^2} - 2r \frac{x_1 x_2}{\sigma_1 \sigma_2} + \frac{x_2^2}{\sigma_2^2} \right)
\]

6-48  
\[
\{x, y < 0\} = \{x < 0, y > 0\} + \{x > 0, y < 0\}
\]

\[
P(x, y < 0) = F_x(0) [1 - F_y(0)] + [1 - F_x(0)] F_y(0)
\]

\[
F_x(0) = 1 - G \left( \frac{n_x}{\sigma_x} \right) \quad F_y(0) = 1 - G \left( \frac{n_y}{\sigma_y} \right)
\]
6.49 If \( \bar{y} = x - y \), then \( E(\bar{y}) = 0 \) \( \sigma^2_{\bar{y}} = \sigma^2_x + \sigma^2_y = 2\sigma^2 \)

Thus, \( \bar{y} \sim N(0;\sigma\sqrt{2}) \) and [see (5.74)]

\[
E(z) = E(|\bar{y}|) = \sqrt{2} \sigma \sqrt{\frac{2}{\pi}} \quad E(z^2) = E(\bar{y}^2) = 2\sigma^2
\]

\[
E(z) = \int \int_D (x-y)f(x,y)dxdy
\]

\[
= \int_{y}^{\infty} \int_{0}^{\infty} (x-y)e^{-x} e^{-y} dxdy = \frac{1}{2}
\]

6.50 Since \( |E(x \bar{y})| \leq E(|x||y|) \), we can assume that the RVs \( x \) and \( y \) are real

(a) \( D \leq E([x \bar{x} - y]^2) = z^2 E(\bar{x}^2) - 2z E(x \bar{y}) + E(y^2) \)

The above is a non-negative quadratic in \( z \) for any \( z \). Hence, its discriminant is non-positive.

(b) Using (a), we obtain

\[
E(\bar{x}^2) + E(y^2) + 2\sqrt{E(x^2)E(y^2)}
\]

\[
\geq E(\bar{x}^2) + E(y^2) + 2 E(x \bar{y}) = E((x \bar{x} + y)^2)
\]

6.52 If \( r_{xy} = 1 \) then

\[
E^2((x - \eta_x)(y - \eta_y)) = E((x - \eta_x)^2)E((y - \eta_y)^2)
\]

i.e., the discriminant of the quadratic

\[
E([z(x - \eta_x) - (y - \eta_y)]^2)
\]

is zero. This is possible only if the quadratic is zero for some \( z = z_0 \). This shows that \( z(x - \eta_x) - (y - \eta_y) = 0 \) in the MS sense.
6-53 If \( E(x) = E(y^2) = E(x y) \), then
\[
E((x - y)^2) = E(x^2) + E(y^2) - 2 E(x y) = 0.
\]
Hence, \( x = y \) in the MS sense.

6-54 If \( x \) has a Cauchy density, then (Prob. 5-31)
\[
E(e^{i\omega x}) = e^{-\alpha |\omega|} \quad E(e^{i\omega kx}) = e^{-\alpha k |\omega|}
\]
Hence, [see (6-240)]
\[
\phi_z(\omega) = E(e^{i\omega x}) = E(E(e^{i\omega nx} | \mathcal{F})) = \sum_{k=0}^{\infty} E(e^{i\omega kx}) e^{-\lambda \frac{k}{k!}} = e^{-\lambda} \sum_{k=0}^{\infty} e^{-\alpha k |\omega|} \frac{k}{k!} = e^{-\lambda} e^{-\alpha |\omega|}
\]

6.55 If \( X = k \), then
\[
Y = n - k
\]
and
\[
Z = X - Y = 2X - n,
\]
where \( Z \) takes the values \(-n, -(n - 2), \ldots, n - 2, n\).
\[
P\{Z = z\} = P\{2X - n = z\} P\{X = \frac{n + z}{2}\}
\]
\[
= \binom{n}{n + z/2} p^{(n+z)/2} q^{(n-z)/2}.
\]
Also
\[
E(Z) = E[2X - n] = 2np - n = n(2p - 1).
\]
\[
\text{Var}(Z) = E[(z - \mu_z)^2] = 4E[(X - np)^2] = 4\text{Var}(X) = 4npq
\]
\[ \phi_X(\omega) = E[e^{j\omega X}] = E[e^{j\omega(aX + bY + c)}] = \phi_X(a\omega) \phi_Y(b\omega)e^{j\omega c} = e^{j\omega(c-(a^2\sigma_1^2+b^2\sigma_2^2))}/2 \]

(see (5-100)).

(b) On comparing with (5-100) we obtain

\[ Z \sim N(c, a^2\sigma_1^2 + b^2\sigma_2^2) \]

(c)

\[ E[Z] = c, \quad \text{Var}(Z) = a^2\sigma_1^2 + b^2\sigma_2^2 \]

6.57

\[ P(X = k|Y = n) = \binom{n}{k} p_1^k q_1^{n-k}, \quad k = 0, 1, 2, \ldots n \]

\[ E[e^{j\omega X}|Y = n] = \sum_{k=0}^{n} e^{j\omega k} P(X = k|Y = n) = (p_1e^{j\omega} + q_1)^n \]

use (5-117). Also

\[ \phi_X(\omega) = E[e^{j\omega X}] = E \left\{ E[e^{j\omega X}|Y = n] \right\} \]

\[ = \sum_{n=0}^{M} E[e^{j\omega X}|Y = n] P(Y = n) \]

\[ = \sum_{n=0}^{\infty} (p_1e^{j\omega} + q_1)^n \binom{M}{n} p_2^n q_2^{M-n} \]

\[ = \sum_{n=0}^{M} \binom{M}{n} [p_2(p_1e^{j\omega} + q_1)]^n q_2^{M-n} \]

\[ = (p_2p_1e^{j\omega} + q_1p_2 + q_2)^M \]

But

\[ 1 - p_1p_2 = 1 - (1 - q_1)(1 - q_2) = q_1p_2 + q_2 \]

Hence

\[ \phi_X(\omega) = \left( pe^{j\omega} + q \right)^M \]

where \( p = p_1p_2 \). Thus

\[ X \sim \text{Binomial}(M, p_1p_2). \]
\[
\int \int f_{XY}(x, y) \, dx \, dy = \int_0^1 \int_x^1 k \, x \, dy \, dx = k \int_0^1 x(1-x) \, dx
\]

\[
k \cdot \frac{1}{6} = 1 \quad \Rightarrow \quad k = 6.
\]

\[
f_X(x) = \int_x^1 6x \, dy = 6x(1-x), \quad 0 < x < 1.
\]

\[
f_Y(y) = \int_0^y 6x \, dy = 3y^2, \quad 0 < y < 1.
\]

\[
E[X] = \int_0^1 x f_X(x) \, dx = 6 \left( \frac{x^3}{3} - \frac{x^4}{4} \right) \bigg|_0^1 = \frac{1}{2}.
\]

\[
E[X^2] = \int_0^1 x^2 f_X(x) \, dx = 6 \left( \frac{x^4}{4} - \frac{x^5}{5} \right) \bigg|_0^1 = \frac{3}{10}.
\]

\[
\text{Var}(X) = \frac{3}{10} - \frac{1}{4} = \frac{1}{20}.
\]

\[
E[Y] = \int_0^1 y f_Y(y) \, dy = 3 \left( \frac{y^4}{4} \right) \bigg|_0^1 = \frac{3}{4}.
\]

\[
E[Y^2] = \int_0^1 y^2 f_Y(y) \, dy = 3 \left( \frac{y^5}{5} \right) \bigg|_0^1 = \frac{3}{5}.
\]

\[
\text{Var}(Y) = \frac{3}{5} - \frac{9}{16} = \frac{3}{80}.
\]

\[
E[XY] = \int \int xy f_{XY}(x, y) \, dy \, dx
\]

\[
= \int_0^1 \int_x^1 xy \, 6x \, dy \, dx = \int_0^1 3x^2(1-x^2) \, dx
\]

\[
= 3 \left( \frac{x^3}{3} - \frac{x^5}{5} \right) \bigg|_0^1 = 3 \left( \frac{1}{3} - \frac{1}{5} \right) = \frac{2}{5}
\]

\[
\text{Cov}(X, Y) = E(XY) - E(X)E(Y)
\]

\[
= \frac{2}{5} - \frac{3}{4} \frac{1}{40}
\]

6.59 (a)

\[
\phi_{X,Y}(\omega_1, \omega_2) = E[e^{j(\omega_1 X + \omega_2 Y)}]
\]

\[
= E[e^{j\omega_1 X}] \cdot E[e^{j\omega_2 Y}] = \phi_X(\omega_1) \cdot \phi_Y(\omega_2)
\]

\[
= e^{j\omega_1 - 1} \cdot e^{j\mu_2 - j\sigma^2_2/2}
\]

(b)

\[
\phi_Z(\omega) = E[e^{j\omega Z}]
\]

\[
= E[e^{j\omega(X+Y)}] = \phi_{X,Y}(\omega, \omega)
\]

\[
= e^{j\lambda(\omega - 1) + (j\mu - j\sigma^2/2)}
\]
6.60 (a) 
\[ Z = \min(X, Y) \]

From Example 6-18, we have
\[ f_Z(z) = 2\lambda e^{-2\lambda z}, \quad z \geq 0 \]

and hence
\[ E[Z] = E[\min(X, Y)] = \frac{1}{2\lambda} \]

(b)
\[ E[\max(2X, Y)] = \int \int \max(2x, y) f_{XY}(x, y) \, dx \, dy \]
\[ = \int \int_{2x \geq y} 2x f_{XY}(x, y) \, dx \, dy + \int \int_{2x < y} y f_{XY}(x, y) \, dx \, dy \]
\[ = \int_0^\infty \int_0^{2x} 2x \lambda^2 e^{-\lambda x} e^{-\lambda y} \, dy \, dx + \int_0^\infty \int_0^{y/2} y \lambda^2 e^{-\lambda x} e^{-\lambda y} \, dx \, dy \]
\[ = \lambda \int_0^\infty 2x e^{-\lambda x} (1 - e^{-2\lambda x}) \, dx + \lambda \int_0^\infty y e^{-\lambda y} (1 - e^{-\lambda y/2}) \, dy \]
\[ = 2\lambda \int_0^\infty \left( xe^{-\lambda x} + 2xe^{-2\lambda x} - 3xe^{-3\lambda x} \right) \, dx \]
\[ = \frac{2}{\lambda} \int_0^\infty \left( u e^{-u} + 2ue^{-2u} - 3ue^{-3u} \right) \, du \]
\[ = \frac{2}{\lambda} \left( 1 + \frac{2}{4} - \frac{3}{9} \right) = \frac{7}{3\lambda}. \]

6.61 (a) 
\[ Z = X - Y \quad \rightarrow \quad -1 < z < 1. \]

\[ F_Z(z) = P(X - Y \leq z) = 1 - P(X - Y > z) \]
\[ = 1 - \int_0^{(1-z)/2} \int_{y+z}^{1-y} f_{XY}(x, y) \, dx \, dy \]
\[ = 1 - \int_0^{(1-z)/2} \left( \int_{y+z}^{1-y} 6xdx \right) \, dy \]
\[ = 1 - 3 \int_0^{(1-z)/2} \{(1 - z^2) - 2(1 + z)y\} \, dy \]
\[ = 1 - \frac{3}{4} (1 + z)(1 - z)^2, \quad z \geq 0. \]

\[ F_Z(z) = P(X - Y \leq z) \]
\[ = \int_0^{(1+z)/2} \int_{x-z}^{1-x} 6xdy \, dx = \int_0^{(1+z)/2} 6x (1 + z - 2x) \, dx \]
\[ = \frac{(1 + z)^3}{4}, \quad z < 0. \]
This gives

\[
f_Z(z) = \begin{cases} 
\frac{3}{4}(1-z)(1+3z), & 0 < z < 1 \\
\frac{3(1+z)^2}{4}, & -1 < z < 0 
\end{cases}
\]

(b) \[ f_X(x) = \int_0^{1-x} 6x \, dy = 6x(1-x), \quad 0 < x < 1 \]

\[
f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{F_X(x)} = \frac{1}{1-x}, \quad 0 < y \leq 1 - x
\]

(c) \[ W = X + Y \]

we have

\[
F_W(w) = P(X + Y \leq w) = \int_0^w \left( \int_0^{w-x} 6x \, dy \right) \, dx = w^3,
\]

and

\[
f_w(w) = \int_0^w 6xdx = 3w^2, \quad 0 < w < 1
\]

\[
E[W] = \frac{3}{4}
\]

\[
E[W^2] = \frac{3}{5}
\]

\[
\text{Var}(X + Y) = \text{Var}(W) = E(W^2) - (E(W))^2 = \frac{3}{5} - \frac{9}{16} = \frac{3}{80}.
\]

6.62

\[
X = \frac{1}{Z}
\]

where \( Z \) represents a Chi-square random variable. Thus (see (4-39))

\[
f_Z(z) = \frac{z^{-1/2}}{\sqrt{2\Gamma(1/2)}} e^{-z/2} = \frac{z^{-1/2}}{\sqrt{2\pi}} e^{-z/2}
\]

or

\[
f_X(x) = \frac{1}{|dx|} f_Z(1/x) = \frac{1}{x^{1/2}} \frac{x^{1/2}}{\sqrt{2\pi}} e^{-1/2x} = \frac{1}{\sqrt{2\pi} x^{3/2}} e^{-1/2x}, \quad x > 0
\]

Also it is given that

\[
f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi}x} e^{-y^2/2x}
\]

so that

\[
f_{XY}(x,y) = f_{Y|X}(y|x) f_X(x) = \frac{1}{2\pi x^2} e^{-(1+y^2)/2x}
\]

and hence

\[
f_Y(y) = \int_0^\infty f_{XY}(x,y) \, dx
\]

\[
= \frac{1}{2\pi} \int_0^\infty \frac{1}{x^2} e^{-(1+y^2)/2x} \, dx
\]

\[
= \frac{1}{2\pi} \frac{2}{1+y^2} \int_0^\infty e^{-u} \, du = \frac{1/\pi}{1+y^2}, \quad -\infty < y < \infty.
\]

Thus \( Y \) represents a Cauchy random variable.
6.63 (a) For any two random variables $X$ and $Y$ we have

$$\sigma^2_{X+Y} = \text{Var}(X + Y) = E[\{(X - \mu_X) + (Y - \mu_Y)\}^2]$$

$$= \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X, Y) = \sigma_X^2 + \sigma_Y^2 + 2\sigma_X\sigma_Y\rho_{XY}$$

$$\leq (\sigma_X + \sigma_Y)^2$$

since $|\rho_{XY}| \leq 1$. Thus

$$\sigma_{X+Y} \leq \sigma_X + \sigma_Y,$$

and hence it easily follows that

$$\frac{\sigma_{X+Y}}{\sigma_X + \sigma_Y} \leq 1.$$ 

(However, (b) is not so easy!)

(b) We shall prove this result in three parts by making use of Holder’s inequality.

(i) **Holder’s inequality:** The function $\log x$ is concave, for $0 < \alpha < 1$, and hence we have

$$\log[\alpha x_1 + (1 - \alpha)x_2] \geq \alpha \log x_1 + (1 - \alpha) \log x_2$$

or

$$x_1^\alpha x_2^{1-\alpha} \leq \alpha x_1 + (1 - \alpha)x_2, \quad 0 < \alpha < 1. \quad (6.63 - 1)$$

Let

$$x_1 = |x|^p, \quad \alpha = \frac{1}{p}, \quad \text{so that} \quad 1 - \alpha = 1 - \frac{1}{p} = \frac{q}{q}, \quad x_2 = |y|^q \quad (6.63 - 2)$$

so that (6.63-1) becomes

$$|xy| \leq \frac{|x|^p}{p} + \frac{|y|^q}{q}, \quad p > 1, \quad (6.63 - 3)$$

the Holder’s inequality. From (6.63-2), note that

$$\frac{1}{p} + \frac{1}{q} = 1, \quad p > 1, \quad q > 1 \quad (6.63 - 4)$$
(ii) Define

$$x = X \left( E\{|X|^p\} \right)^{-1/p}, \quad y = Y \left( E\{|Y|^q\} \right)^{-1/q}$$

where \( p \) and \( q \) are as in (6.63-4). Substituting these into the Holder’s inequality in (6.63-3), we get

$$|XY| \leq p^{-1} |X|^p \left( E\{|X|^p\} \right)^{1/p-1} \left( E\{|Y|^q\} \right)^{1/q} + q^{-1} |Y|^q \left( E\{|Y|^q\} \right)^{1/q-1} \left( E\{|X|^p\} \right)^{1/p}. \quad (6.63 - 5)$$

Taking expected values on both sides of (6.63-5), we get

$$E\{|XY|\} \leq \left( E\{|X|^p\} \right)^{1/p} \left( E\{|Y|^q\} \right)^{1/q} \quad (6.63 - 6)$$

which represents the generalization of the Cauchy-Schwarz inequality. (Note \( p = q = 2 \) corresponds to Cauchy-Schwarz inequality)

(iii) To prove the desired inequality, notice that

$$|X + Y|^p = |X + Y||X + Y|^{p-1} \leq |X||X + Y|^{p-1} + |Y||X + Y|^{p-1}, \quad p > 1$$

and taking expected values on both sides we get

$$E\{|X + Y|^p\} \leq E\{|X||X + Y|^{p-1}\} + E\{|Y||X + Y|^{p-1}\}. \quad (6.63 - 7)$$

Applying (6.63-6) to each term on the right side of (6.63-7) we get

$$E\{|X||X + Y|^{p-1}\} \leq \left( E\{|X|^p\} \right)^{1/p} \left( E\{|X + Y|^{(p-1)q}\} \right)^{1/q} \quad (6.63 - 8)$$

and

$$E\{|Y||X + Y|^{p-1}\} \leq \left( E\{|Y|^q\} \right)^{1/p} \left( E\{|X + Y|^{(p-1)q}\} \right)^{1/q} \quad (6.63 - 9)$$

Using (6.63-8) and (6.63-9) together with \((p - 1)q = p\) in (6.63-7) we get

$$E\{|X + Y|^p\} \leq \left[ \left( E\{|X|^p\} \right)^{1/p} \left( E\{|Y|^q\} \right)^{1/p} \right] \left( E\{|X + Y|^p\} \right)^{1/q} \quad (6.63 - 10)$$

or for \( p > 1 \)

$$\left( E\{|X + Y|^p\} \right)^{1/p} \leq \left( E\{|X|^p\} \right)^{1/p} + \left( E\{|Y|^q\} \right)^{1/p}.$$

the desired inequality. Since \( p = 1 \) follows trivially, we get

$$\frac{\left( E\{|X + Y|^p\} \right)^{1/p}}{\left( E\{|X|^p\} \right)^{1/p} + \left( E\{|Y|^q\} \right)^{1/p}} \leq 1, \quad p \geq 1.$$
6.64 (a) See Example 6-41. From there

\[ E(Y|X = x) = \mu_Y + \frac{\rho_{XY}\sigma_Y(x - \mu_X)}{\sigma_X} \]

(b) Similarly

\[ f_{X|Y}(X|Y = y) \sim N(\mu, \sigma^2) \]

where

\[ \mu = \mu_X + \frac{\rho_{XY}\sigma_X(y - \mu_Y)}{\sigma_Y} \]

and

\[ \sigma^2 = \sigma_X^2 (1 - \rho_{XY}^2). \]

Since

\[ E(X^2|Y = y) = \text{Var}(X|Y = y) + (E[X|Y = y])^2 \]

we obtain

\[ E(X^2|Y = y) = \sigma^2 + \mu^2 \]

6.65 (a) See footnote 4, Chapter 8, Page 337. From there (or directly) we have

\[ \text{Var}(X|Y) \triangleq E(X^2|Y) - (E\{X|Y\})^2 \]

\[ \text{Var}(E\{X|Y\}) \triangleq E[E\{X|Y\}]^2 - (E[E\{X|Y\}])^2 \]

so that

\[ E[\text{Var}(X|Y)] + \text{Var}(E\{X|Y\}) = E[E\{X^2|Y\}] - (E[E\{X|Y\}])^2 \]

\[ = E(X^2) - [E(X)]^2 = \text{Var}(X) \quad (1) \]

or

\[ \text{Var}(X) \geq E[\text{Var}\{X|Y\}] \]

Also

\[ \text{Var}(X) \geq \text{Var}[E\{X|Y\}] \]

(b) See (1).
6.66

\[ Z = aX + (1 - a)Y, \quad 0 < a < 1 \]
\[ \sigma_Z^2 = \text{Var}(Z) = a^2 \sigma_1^2 + (1 - a)^2 \sigma_2^2 \]
\[ \frac{\partial \sigma_Z^2}{\partial a} = 2a \sigma_1^2 + 2(1 - a)(-1) \sigma_2^2 = 0 \]

or

\[ a(\sigma_1^2 + \sigma_2^2) = \sigma_2^2 \]
\[ a = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2} < 1 \]

minimizes \text{Var}(Z).

6.67 From (6-240)

\[ \mathbb{E}\{g(\bar{x}, y)\} = \mathbb{E}\{\mathbb{E}[g(\bar{x}, y) | y]\} = \mathbb{E}\{g(x_n, y)\mathbb{P}(\bar{x} = x_n)\} . \]

From (4-74) with \( A_n = \{\bar{x} = x_n\} \)

\[ f_z(z) = \sum_{n} f_z(z | x = x_n) \mathbb{P}(x = x_n) \]

6.68 (a) The conditional density \( f(\bar{y} | x) \) is \( N(\bar{r}x; \sigma \sqrt{1-r^2}) \) [see (7-42)]. Hence

\[ \mathbb{E}(f_y(\bar{y} | x)) = \int_{-\infty}^{\infty} f_y(y | x)f_y(y)dy \]

\[ = \frac{1}{2\pi \sigma^2 \sqrt{1-r^2}} \int_{-\infty}^{\infty} \exp \left\{ \frac{-(y-\bar{r}x)^2}{2\sigma^2(1-r^2)} \right\} \exp \left\{ \frac{-y^2}{2\sigma^2} \right\} dy \]
\[ = \frac{1}{\sigma \sqrt{2\pi(2-r^2)}} \exp \left\{ \frac{-r^2x^2}{2\sigma^2(2-r^2)} \right\} \]

(b) From (6-241) it follows that

\[ \mathbb{E}(f_y(\bar{y} | x)f_x(x)) = \mathbb{E}(f_y(\bar{y} | x)E(f_y(\bar{y} | x))) = \int_{-\infty}^{\infty} f_x(x) \mathbb{E}(f_y(y | x))f_x(x)dx \]

\[ = \frac{1}{2\pi \sigma^2 \sqrt{2\pi(2-r^2)}} \int_{-\infty}^{\infty} \exp \left\{ -\frac{x^2}{2\sigma^2} \right\} \exp \left\{ \frac{-r^2x^2}{2\sigma^2(2-r^2)} \right\} dx = \frac{1}{2\pi \sigma^2 \sqrt{4-r^2}} \]
We shall use (6-64) and Price's theorem (10-94):

\[
\frac{\partial E(|xy|)}{\partial \mu} = E \left\{ \frac{d|x|}{dx} \frac{d|y|}{\partial y} \right\} = E(\text{sgn} x \text{sgn} y)
\]

\[
P(\omega > 0) - P(\omega < 0) = \frac{2\alpha}{\pi} = \frac{2}{\pi} \arcsin \frac{\mu}{\sigma_1 \sigma_2}
\]

If \( \mu = 0 \), then the RVs \( x \) and \( y \) are independent, hence,

\[
E(|x| E(|y|) = \frac{2}{\pi} \sigma_1 \sigma_2
\]

[see (5-74)]. Integrating (1) and using the above, we obtain

\[
E(|x| y|) = \frac{2}{\pi} \int_0^\mu \arcsin \frac{c}{\sigma_1 \sigma_2} \, dc + \frac{2}{\pi} \sigma_1 \sigma_2 = \frac{2\sigma_1 \sigma_2}{\pi} (\cos \alpha + \alpha \sin \alpha)
\]

6-70 From Example 6-41

\[
f(y|x) = N(\eta_2 + \frac{\sigma_2}{\sigma_1} x; \sigma_2 \sqrt{1 - r^2}) = N(4 + x; \sqrt{3})
\]

\[
f(x|y) = N(\eta_1 + \frac{\sigma_1}{\sigma_2} y; \sigma_1 \sqrt{1 - r^2}) = N(3 + \frac{y}{4}; \sqrt{3}/2)
\]

6-71 The mass density in the square \(|x| \leq 1, |y| \leq 1\) of the xy plane equals 1/4; hence, \(P_{r \leq 1} = \pi/4\) and \(P_{r \leq r} = \pi r^2/4\) for \(r < 1\). This yields

\[
P(r \leq r, r \leq 1) = \begin{cases} 
\frac{P(r \leq r) - \pi r^2/4}{P(M)} & r \leq 1 \\
P(r \leq 1) - \pi/4 & r > 1
\end{cases}
\]

\[
F_r(r|\mu) = \frac{P(r \leq r, \mu)}{P(M)} = \begin{cases} 
r^2 & r \leq 1 \\
1 & r > 1
\end{cases}
\]

\[
f_r(r|m) = \begin{cases} 
2r & r < 1 \\
0 & \text{otherwise}
\end{cases}
\]
\[ z = x + y \quad \quad w = x \quad \quad f_{xz}(x,z) = f_{xy}(x, z-x) \]

If \( f_{xy}(x,y) = f_x(x)f_y(y) \), then

\[ f_z(z|x) = \frac{f_{xz}(x,z)}{f_x(x)} = f_y(z-x) \]

6-73 The system \( z = F_x(x) \quad w = F_y(y|x) \) has a solution only if \( z \leq z \leq 1 \) and \( 0 \leq w \leq 1 \). Furthermore,

\[ \frac{\partial z}{\partial x} = f_x(x) \quad \quad \frac{\partial z}{\partial y} = 0 \]

\[ J = f_x(x)f_y(y|x) \]

\[ \frac{\partial w}{\partial x} = f_y(y|x) \]

\[ f_{zw}(z,w) = \frac{f_{xy}(x,y)}{f_x(x)f_y(y|x)} = 1 \quad \text{for} \quad 0 \leq z, w \leq 1 \]

6-74 We introduce the events \( C_r = \{ \text{we selected the rth coin} \} \) and \( A_k = \{ \text{heads in a specific order} \} \). From the assumptions it follows that

\[ P(C_r) = \frac{1}{m} \quad P(A_k|C_r) = p_r^k(1-p_r)^{n-k} \]

We wish to find the probability \( P(C_r|A_k) \). The events \( C_r \) form a partition; hence,

\[ P(C_r|A_k) = \frac{\frac{1}{m} P(a_k|C_r)}{\frac{1}{m} \sum_{r=1}^{m} P(A_k|C_r)} \]
6.75 We wish to show that

\[ E(x^2) = \frac{n}{n-1} \]

From page 207: \( x^2 = \frac{y^2}{z} \) where \( y \) is \( N(0,1) \) and \( z \) is \( \chi^2(n) \). Hence, \( E(y^2) = 1 \) and (also (4-35) and (4-39))

\[
E \left\{ \frac{1}{z} \right\} = \frac{1}{2^{n/2} \Gamma(n/2)} \int_0^\infty z^{n/2-2}e^{-z/2}dz = \frac{2^{n/2-1}\Gamma(n/2-1)}{2^{n/2} \Gamma(n/2)}
\]

From this and the independence of \( y \) and \( z \) it follows that

\[ E(x^2) = n \ E(y^2) \ E \left\{ \frac{1}{z} \right\} = \frac{n}{n-2} \]

6.76 From (6-222):

\[ R_x(x) = \exp \left\{ -\int_0^x \beta_x(t)dt \right\} = \exp \left\{ -k \int_0^x \beta_x(t)dt \right\} = R_y^k(t) \]

6.77 From (5-89) it follows with \( z = |z|^2 \) and \( a = \varepsilon^2 \) that

\[ E(|z|^2 > \varepsilon^2) \leq \frac{E(|z|^2)}{\varepsilon^2} \]

for any \( z \). And the result follows with \( z = x - y \).
6-78 \[ E[U(a-x)] = \int_{-\infty}^{\infty} U(a-x)f(x)dx = \int_{-\infty}^{a} f(x)dx = F_x(a) \]

\[ E[U(b-y)] = F_y(b) \]

\[ E[U(a-x)U(b-y)] = \int_{-\infty}^{a} \int_{-\infty}^{b} f(x,y)dxdy = F_{xy}(a,b) \]

Hence

\[ F_{xy}(a,b) = F_x(a)F_y(b) \]

6-79 From Example 6-38

\[ E(y|x \leq 0) = \int_{-\infty}^{\infty} y f_y(y|x \leq 0)dy = \frac{1}{F_x(0)} \int_{-\infty}^{\infty} y \frac{\partial F(0,y)}{\partial y} dy \]

From (7-41) and (7-57)

\[ \int_{-\infty}^{\infty} E(y|x)f_x(x)dx = \int_{-\infty}^{0} y \int_{-\infty}^{0} f(x,y)dxdy = \int_{-\infty}^{\infty} y \frac{\partial F(0,y)}{\partial y} dy \]
7-1 \[ 0 \leq P(x_1 < x \leq x_2, y_1 < y \leq y_2, z_1 < z \leq z_2) = \]
\[= P(x \leq x_2, y_1 < y \leq y_2, z_1 < z \leq z_2) - P(x \leq x_2, y_1 < y \leq y_2, z_1 < z \leq z_2) = \]
\[= P(x \leq x_2, y \leq y_2, z_1 < z \leq z_2) - P(x \leq x_2, y \leq y_1, z_1 < z \leq z_2) = \]
\[= P(x \leq x_1, y \leq y_2, z_1 < z \leq z_2) + P(x \leq x_1, y \leq y_1, z_1 < z \leq z_2) = \]
\[= P(x \leq x_2, y \leq y_2, z \leq z_1) - P(x \leq x_2, y \leq y_2, z \leq z_1) = \]
\[= P(x \leq x_1, y \leq y_2, z \leq z_1) + P(x \leq x_1, y \leq y_1, z \leq z_1) = \]
\[= P(x \leq x_1, y \leq y_2, z \leq z_1) + P(x \leq x_1, y \leq y_2, z \leq z_1) + P(x \leq x_1, y \leq y_1, z \leq z_1) \]
\[+ P(x \leq x_1, y \leq y_1, z \leq z_1) - P(x \leq x_1, y \leq y_1, z \leq z_1) \]

7-2 \[ P\{x_A = 1, x_B = 1, x_C = 1\} = P(ABC) = 1/4 \]
\[P\{x_A = 1\} = P(A) = 1/2 \quad P\{x_B = 1\} = P(B) = 1/2 \]
\[P\{x_C = 1\} = P(C) = 1/2 \text{ hence} \]
\[P\{x_A = 1, x_B = 1, x_C = 1\} \neq P\{x_A = 1\}P\{x_B = 1\}P\{x_C = 1\} \]
\text{hence } x_A, x_B, x_C \text{ are not independent. But}
\[P\{x_A = 1, x_B = 1\} = P(AB) = 1/4 = P\{x_A = 1\}P\{x_B = 1\} \]
\text{Similarly for any other combination, e.g.,}
\[\text{Since } P(A) = P(AB) + P(A\overline{B}), \text{ we conclude that} \]
\[P(AB) = 1/2 - 1/4 = 1/4 \quad P(\overline{B}) = 1 - P(B) = 1/2 \]
\[P\{x_A = 1, x_B = 0\} = P(AB) = 1/4 \]
\[P\{x_B = 0\} = P(\overline{B}) = 1/2 \text{ hence} \]
\[P\{x_A = 1, x_B = 0\} = P\{x_A = 1\}P\{x_B = 0\} \]
7-3 If \( x, y, z \) are independent in pairs, then

\[
\text{r}_{xy} = \text{r}_{xz} = \text{r}_{yz} = 0
\]

and (7-60) yields (we assume \( \eta_x = \eta_y = \eta_z = 0 \))

\[
\Phi(\omega_1, \omega_2, \omega_3) = \exp \left\{ -\frac{1}{2} \left( \sigma_{1\omega_1}^2 + \sigma_{2\omega_2}^2 + \sigma_{3\omega_3}^2 \right) \right\}
\]

\[
f(x_1, x_2, x_3) = f(x_1)f(x_2)f(x_3)
\]

7-4 \( x = x_1 + x_2 + x_3 \). To determine \( E(x^4) \) we shall use char. functions

\[
\tau(\omega) = \int_{-1/2}^{1/2} e^{i\omega x} dx = \frac{2 \sin (\omega/2)}{\omega}
\]

\[
\tau'(\omega) = \left[ \frac{2 \sin (\omega/2)}{\omega} \right]^3 = \left( 1 - \frac{\omega^2}{24} + \frac{\omega^4}{1920} - \cdots \right)^3
\]

The coefficient of \( \omega^4 \) in this expansion equals

\[
\frac{13}{1920} \text{ hence } \frac{1}{4!} \frac{d^4 \tau(0)}{d\omega^4} = \frac{13}{1920}
\]

and [see (5-103)]

\[
E(x^4) = m_4 = \frac{13 \times 41}{1920} = \frac{13}{80}
\]
7-5 (a) The joint density \( f(x, y) \) has circular symmetry because

\[
f(x, y) = \int_{-\infty}^{\infty} f(rx^2 + y^2 + z^2) \, dz
\]

depends only on \( x^2 + y^2 \). The same holds for \( f(x, z) \) and \( f(y, z) \). And since the RVs \( x, y, \) and \( z \) are independent, they must be normal [see (6-29)].

(b) From (a) it follows that the RVs \( v_x, v_y, v_z \) are \( N(0; \sqrt{kT/m}) \).

With \( \sigma^2 = kT/m \) and \( n = 3 \) it follows from (7-62) - (7-63) and (5-25) that

\[
f_v(v) = \sqrt{\frac{2m^3}{\pi k T^3}} \, v^2 \, e^{-mv^2/2kT} \, U(v)
\]

\[
E(v) = 2\sqrt{\frac{2kT}{nm}} \quad E(v^2) = 1 \times 3 \cdots (2n+1) \left( \frac{kT}{m} \right)^n
\]

7-6 From Prob. 6-52: \( y = ax + b, \ z = cy + d, \) hence,

\[
z = A \eta + B \quad \eta_z = A \eta_x + B \quad \sigma_z = A \sigma_x
\]

\[
E((z - \eta_z)(x - \eta_x)) = E(A(x - \eta_x)(x - \eta_x)) = A \sigma_x^2 = \sigma_x \sigma_z
\]

7-7 It follows from (6-241) with \( g_1(x) = x, \ g_2(y) = y \) if we replace all densities with conditional densities assuming \( x_3 \).
7-8 Reasoning as in (7-82), we conclude that
\[ \mathbb{E}[(y - (a_1x_1 + a_2x_2))^2] \text{ is minimum if} \]
\[ \mathbb{E}[(y - (a_1x_1 + a_2x_2))x_i] = 0 \quad i = 1, 2 \]

With \( R_{01} = \mathbb{E}(y|x_1) \), \( R_{1j} = \mathbb{E}(x_ix_j) \), the above yields
\[ R_{01} = a_1R_{11} + a_2R_{12} \quad R_{02} = a_1R_{12} + a_2R_{22} \]

But \( \hat{\mathbb{E}}(y|x_1) = A x_1 \)
\( A = \frac{R_{01}}{R_{11}} = a_1 + a_2 \frac{R_{12}}{R_{11}} \)
\[ \hat{\mathbb{E}}(\hat{\mathbb{E}}(y|x_1, x_2)|x_1) = \hat{\mathbb{E}}(a_1x_1 + a_2x_2|x_1) \]
\[ = a_1x_1 + a_2 \hat{\mathbb{E}}(x_2|x_1) = \left( a_1 + a_2 \frac{R_{12}}{R_{11}} \right) x_1 = A x_1 \]

7-9 As in Prob. 6-51
\[ \mathbb{E}^2(x_ix_j) \leq \mathbb{E}^2(x_i)\mathbb{E}^2(x_j) = M^2 \quad |\mathbb{E}(x_i x_j)| \leq M \]

\[ \mathbb{E}(s^2|n = n) = \mathbb{E}\left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{x_i x_j}{n} \right\} \leq \frac{n}{M} \]

Hence [see (6-240)]
\[ \mathbb{E}(s^2) = \mathbb{E}(\mathbb{E}(s^2|n)) < \mathbb{E}(M^2) \]
7-10 As we know,

\[ 1 + x + \cdots + x^n + \cdots = \frac{1}{1 - x} \quad |x| < 1 \]

Differentiating, we obtain

\[ 1 + 2x + \cdots + nx^{n-1} + \cdots = \sum_{k=1}^{\infty} kx^{k-1} = \frac{1}{(1-x)^2} \quad \text{(i)} \]

The RV \( X_1 \) equals the number of tosses until heads shows for the first time, Hence, \( X_1 \) takes the values 1, 2, \ldots with \( P(X_1 = k) = pq^{k-1} \). Hence, [see (3-12) and (i)]

\[ E(X_1) = \sum_{k=1}^{\infty} k p \cdot q^{k-1} = \frac{p}{(1-q)^2} = \frac{1}{p} \]

Starting the count after the first head shows, we conclude that \( \text{RV} \ X_2 - X_1 \) has the same statistics as the RV \( X_1 \). Hence,

\[ E(X_2 - X_1) = E(X_1) \quad E(X_2) = 2E(X_1) = \frac{2}{p} \]

Reasoning similarly, we conclude that

\[ E(X_n - X_{n-1}) = E(X_1). \quad \text{Hence (induction)} \]

\[ E(X_n) = E(X_{n-1}) + E(X_1) = \frac{n-1}{p} + \frac{1}{p} = \frac{n}{p} \]

7-11 If \( n \) accidents occur in a day, the probability that \( \omega \) of them will be fatal equals \( \binom{n}{m} p^m q^{n-m} \) for \( m \leq n \) and zero for \( m > n \). Hence,

\[ P(\omega = m \mid n = n) = \begin{cases} 0 & \quad m > n \\ \binom{n}{m} p^m q^{n-m} & \quad m \leq n \end{cases} \]

This yields

\[ E(e^{i\omega} \mid n = n) = \sum_{m=0}^{n} e^{i\omega} \binom{n}{m} p^m q^{n-m} = (p e^{i\omega} + q)^n \]

But

\[ P(\omega = n) = e^{-a} \frac{a^n}{n!} \quad n = 0, 1, \ldots \]
Hence,
\[
E(e^{i\omega \mathbf{w}}) = E\{E(e^{i\omega \mathbf{w}} | \mathbf{n})\} = E\{(p e^{i\omega} + q) 0\}
\]
\[
\sum_{n=0}^{\infty} (p e^{i\omega} + q)^n e^{-a} \frac{a^n}{n!} = e^{a(p e^{i\omega} + q) e^{-a} = e^{ap(e^{i\omega} - 1)}
\]

This shows that the RV \( \mathbf{n} \) is Poisson distributed with parameter \( a p \) [see (5-119)].

7-12 We shall determine first the conditional distribution
\[
F_s(s|n = n) = \frac{P(s \leq s, n = n)}{P(n = n)}
\]
The event \( \{s < s, n = n\} \) consists of all outcomes such that \( n = n \) and \( \sum_{k=1}^{n} x_k \leq s \). Since the RV \( n \) is independent of the RVs \( x_k \), this yields
\[
F_s(s|n = n) = P(\sum_{k=1}^{n} x_k \leq s)P(n = n)/P(n = n)
\]

From the above and the independence of the RVs \( x_k \) it follows that [see (7-51)]
\[
f_s(s|n = n) = f_1(s) * f_2(s) * \cdots * f_n(s)
\]

Setting \( A_k = \{n = k\} \) in (4-74), we obtain
\[
f_s(s) = \sum_{k} p_k \left[f_1(s) * \cdots * f_k(s)\right]
\]
7-13 From the independence of the RVs \( n \) and \( X_i \) it follows that

\[
E\{e^{s\sum X_i} \mid n = k \} = E\left( e^{s(x_1 + \cdots + x_k)} \right) \\
= E(\sum_{i=1}^{s} e^{sX_i}) = \sum_{i=1}^{s} E(e^{sX_i}) = \tilde{\varphi}_x(s)
\]

Hence,

\[
\tilde{\varphi}_y(s) = E(e^{sY}) = E(E(e^{sY} \mid n)) = E(\tilde{\varphi}_x^n(s)) \\
= \Gamma_n [\varphi_x(s)] \text{ because } E(z^n) = \frac{1}{n}
\]

**Special case.** If \( n \) is Poisson with parameter \( a \), then [see (5-119)]

\[
\Gamma_n(z) = e^{az - a} \\
\tilde{\varphi}_y(s) = e^{x(s)} - a
\]

7-14

\[
\begin{align*}
F_{X_i}(x) & \\
\{y \leq y\} & = \{x_1 \leq y, x_2 \leq y, \ldots, x_n \leq y\} \\
\text{From the independence of } X_i \text{ and the above it follows that} \\
F_y(y) & = P(y \leq y) = P(x_1 \leq y) \cdots P(x_n \leq y) \\
& = F_1(y) \cdots F_n(y)
\end{align*}
\]

where \( F_i(y) = y \) for \( 0 \leq y \leq 1 \).
The RV $x$ is defined in the space $S$. The set

\[ C = \{ z < z \sim z + dz, \ w < w \leq w + dz \} \quad z > w \]

is an event in the space $S_n$ of repeated trials and its probability equals

\[ P(C) = f_{zw}(z,w)dzdw \]

We introduce the events

\[ D_1 = \{ x \leq w \} \quad D_2 = \{ w < x \leq w + dw \} \quad D_3 = \{ w + dw < x \leq z \} \]

\[ D_4 = \{ z < x \leq z + dz \} \quad D_5 = \{ z + dz < x \} \]

These events form a partition of $S$ and their probabilities $p_i = P(D_i)$ equal

\[ F_x(w) \quad f_x(w)dw \quad F_x(z)-F_x(w+dw) \quad f_x(z)dz \quad 1-F_x(z+dz) \]

respectively. The event $C$ occurs iff the smallest of the RVs $x_1$ is in the interval $(w, w+dw)$, the largest is in the interval $(z, z+dz)$, and, consequently, all others are between $w+dw$ and $z$. This is the case iff $D_1$ does not occur at all, $D_2$ occurs once, $D_3$ occurs $n-2$ times, $D_4$ occurs once, and $D_5$ does not occur at all. With

\[ k_1=0 \quad k_2=1 \quad k_3=n-2 \quad k_4=1 \quad k_5=0 \]

it follows from (4-102) that

\[ P(C) = \frac{n!}{(n-2)!} p_2 p_3^{n-2} p_4 = n(n-1)f_x(w)dw [f_x(z)-F_x(w+dw)]^{n-1}f_x(z)dz \]

for $z > w$, and 0 otherwise.

7-16 If $z$ is $N(\eta, 1)$ then

\[ E(e^{z^2}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{z^2}e^{-(s-\eta)^2/2}dz \]
\[ s z^2 - \frac{(z-\eta)^2}{2} = \left( s - \frac{1}{2} \right) \left( z - \frac{\eta}{1-2s} \right)^2 + \frac{\eta^2 s}{1-1s} \]

Since

\[ \frac{1}{\sqrt{2\pi}} \int_{-\eta}^{\infty} e^{-a(z-b)^2} dz = \frac{1}{\sqrt{2a}} \]

the above yields

\[ E(e^{sz^2}) = \frac{1}{\sqrt{2(1/2-S)}} \exp \left\{ \frac{\eta^2 S}{1-2S} \right\} \]

\[ \Phi_w(s) = \frac{1}{\sqrt{1-2s}} \exp \left\{ \frac{\eta_1 s}{1-2s} \right\} \ldots \frac{1}{\sqrt{1-2s}} \exp \left\{ \frac{\eta_n s}{1-2s} \right\} \]

7-17 We wish to show that the RVs

\[ \tilde{x} = \frac{1}{n} \sum_{i=1}^{n} x_i \quad s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \tilde{x})^2 \]

are independent. Since \( s^2 \) is a function of the \( n \) RVs \( x_i - \tilde{x} \), it suffices to show that each of these RVs is independent of \( \tilde{x} \). We assume for simplicity that \( E(x_i) = 0 \). Clearly,

\[ E(x_i \tilde{x}) = \frac{1}{n} E \{ x_i^2 \} = \frac{\sigma^2}{n} \quad E(\tilde{x}^2) = \frac{1}{n-1} \sum_{i=1}^{n} x_i^2 = \frac{\sigma^2}{n} \]

because \( E(x_i x_j) = 0 \) for \( i \neq j \). Hence,

\[ E((x_i - \tilde{x}) \tilde{x}) = 0 \]

Thus, the RVs \( x_i - \tilde{x} \) and \( \tilde{x} \) are orthogonal; and since they are jointly normal, they are independent.

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7-18 Since \( \eta_s = \alpha_0 + \alpha_1 \eta_1 + \alpha_2 \eta_2 \) [see (7-87)], the mean of the error

\[
epsilon = s - (\alpha_0 + \alpha_1 \eta_1 + \alpha_2 \eta_2) = (s - \eta_s) - [\alpha_1 (x_1 - \eta_1) + \alpha_2 (x_2 - \eta_2)]
\]

is zero. Furthermore, \( \epsilon \) is orthogonal to \( x_1 \), hence, it is also orthogonal to \( x_1 - \eta_1 \).

7-19 From the orthogonality principle:

\[
\hat{E}(y|x_1, x_2) = a_1 x_1 + a_2 x_2 \quad \quad y - (a_1 x_1 + a_2 x_2) \perp x_1, x_2
\]

\[
\hat{E}(y|x_1) = A x_1 \quad \quad y - A x_1 \perp x_1
\]

Hence

\[
y - (a_1 x_1 + a_2 x_2) - (y - A x_1) = a_1 x_1 + a_2 x_2 - A x_1 \perp x_1
\]

From this it follows that

\[
\hat{E}(a_1 x_1 + a_2 x_2 | x_1) = A x_1
\]

\[
\hat{E}(E(y|x_1, x_2) | x_1) = E(y|x_1)
\]
The event \( \{x \leq x\} \) occurs if there is at least one point in the interval \((0,x)\); the event \( \{y \leq y\} \) occurs if all the points are in the interval \((0,y)\):

\[
A_x = \{\text{at least one point in } (0,x)\} = \{x \leq x\}
\]

\[
B_y = \{\text{no points in } (y,1)\} = \{\text{all points in } (0,y)\} = \{y \leq y\}
\]

Hence, for \(0 \leq x \leq 1\), \(0 \leq y \leq 1\)

\[
F_x(x) = P(A_x) = 1 - P(\overline{A_x}) = 1 - (1 - x)^n
\]

\[
F_y(y) = P(B_y) = y^n
\]

Furthermore,

\[
\{x \leq x, y \leq y\} = A_x \cap B_y
\]

\[
A_x \cap B_y + \overline{A_x} \cap B_y = B_y
\]

If \(x \leq y\) then

\[
\overline{A_x \cap B_y} = \{\text{all points in } (x,y)\}
\]

\[
P(\overline{A_x \cap B_y}) = (y - x)^n
\]

If \(x > y\), then \(\overline{A_x \cap B_y} = \emptyset\). Hence

\[
F_{xy}(x,y) = P(A_x \cap B_y) = \begin{cases} y^n - (y - x)^n & \text{if } x \leq y \\ y^n & \text{if } x > y \end{cases}
\]
Suppose that $E(x_{1}) = 0$, $E(x_{2}^{2}) = \sigma^{2}$, $E(x_{4}^{4}) = \mu_{4}$

If $A = \sum_{i=1}^{n} x_{i}^{2}$, then $E(A) = n \sigma^{2}$

$E(A^{2}) = \sum_{i,j=1}^{n} E(x_{i}^{2}x_{j}^{2}) = n \mu_{4} + (n^{2} - n) \sigma^{4}$

because

$$E(x_{i}^{2}x_{j}^{2}) = \begin{cases} \mu_{4} & i = j \\ \sigma^{4} & i \neq j \end{cases}$$

Furthermore

$$E(\bar{x}_{i}^{2}) = \frac{1}{n^{2}} \sum_{i=1}^{n} E(x_{i}^{2}) = \frac{1}{n^{2}} [n \mu_{4} + (n - 1) \sigma^{4}]$$

$$E(\bar{x}_{j}^{2}) = \frac{1}{n} [\mu_{4} + (n - 1) \sigma^{4}]$$

$$E(\bar{x}^{4}) = \frac{1}{n^{4}} E\left(\left(\sum_{i=1}^{n} x_{i}\right)^{4}\right) = \frac{1}{n^{4}} [n \mu_{4} + 3n(n - 1) \sigma^{4}]$$

because

$$E(x_{j}^{4}x_{k}^{4}x_{r}^{4}) = \begin{cases} \mu_{4} & i = j = k = r \quad [n \text{ such terms}] \\ \sigma^{4} & i = j \neq k = r \quad [3n(n-1) \text{ such terms}] \\ 0 & \text{otherwise} \end{cases}$$

Clearly, $(n - 1) \bar{\nu} = \sum_{i=1}^{n} (x_{i}^{2} - \bar{x}^{2})^{2} = A - n \bar{x}^{2}$, $E(\bar{\nu}) = \sigma^{2}$. Hence

$$(n - 1)^{2} E(\bar{\nu}^{2}) = E(A^{2}) - 2n E(\bar{x}^{2}) + n^{2} E(x^{4})$$

$$= n \mu_{4} + (n^{2} - n) \sigma^{4} - 2[\mu_{4} + (n - 1) \sigma^{4}] + \frac{1}{n} [\mu_{4} + 3(n - 1) \sigma^{4}]$$

This yields

$$E(\bar{\nu}^{2}) = \frac{\mu_{4}}{n} + \frac{n^{2} - 2n + 3}{n(n - 1)} \sigma^{4} = \sigma^{4} + \frac{\sigma^{2}}{\bar{\nu}}$$

**Note** If the RVs $x_{i}$ are $N(0, \sigma^{2})$, then $\mu_{4} = 3 \sigma^{4}$

$$\sigma_{\bar{\nu}}^{2} = \frac{1}{n} (3 \sigma^{4} - \frac{n - 3}{n - 1} \sigma^{4}) = \frac{2}{n - 1} \sigma^{4}$$
From Prob. 6-49:

\[ \text{E}(|x_{2i} - x_{2i-1}|) = \frac{2\sigma}{\sqrt{\pi}} \]

\[ \text{E}(|x_{2i} - x_{2i-1}|^2) = 2\sigma^2 \]

Hence,

\[ \text{E}(|x_{2i} - x_{2i-1} \parallel x_{2j} - x_{2j-1}|) = \begin{cases} 2\sigma^2 & \text{if } i = j \\ 4\sigma^2/\pi & \text{if } i \neq j \end{cases} \]

\[ \text{E}(z) = \frac{\sqrt{\pi}}{2n} \frac{2\sigma n}{\sqrt{\pi}} = \sigma \]

\[ \text{E}(z^2) = \frac{n}{4n} \left[ 2n\sigma^2 + \frac{4\sigma^2}{\pi} (n^2 - n) \right] \]

\[ \frac{\sigma^2}{z} = \frac{\pi}{2n} \sigma^2 + \left( 1 - \frac{1}{n} \right) \sigma^2 - \sigma^2 = \frac{n-2}{2n} \sigma^2 \]

Then \[ \sum_{j} a_{ij} R_{ji} = 1 \]

Hence,

\[ \text{E}(XR^{-1}X^t) = \text{E}(\sum_{i=1}^{n} \sum_{j=1}^{n} x_i a_{ij} x_j) \]

\[ = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} R_{ji} = \sum_{i=1}^{n} 1 = n \]

The density \( f(z) \) of the sum \( z = x_1 + \cdots + x_n \) tends to a normal curve with variance \( \sigma^2 + \cdots + \sigma^2 \to \infty \) as \( n \to \infty \) (we assume \( \sigma_i \to c > 0 \)). Hence, \( f_z(z) \) tends to a constant in any interval of length \( 2\pi \). The result follows as in (5-37) and Prob. 5-20.
7-25 Since $a_n - a \to 0$, we conclude that

$$E((x_n - a)^2) = E((x_n - a_n) + (a_n - a))^2)$$

$$= E((x_n - a_n)^2) + 2(a_n - a)E(x_n - a_n) + (a_n - a)^2 + 0$$

as $n \to \infty$.

7-26 If $E(x_{n-m}^{-}) \to a$ as $n, m \to \infty$, then, given $\epsilon > 0$, we can find a number $n_0$ such that

$$E(x_{n-m}) = a + \theta(n, m)$$

$$|\theta| < \epsilon$$

if $n, m > 0$.

Hence,

$$E((x_n - x_m)^2) = E(x_n^2) + E(x_m^2) - 2E(x_n x_m)$$

$$= a + \theta_1 + a + \theta_2 - 2(a + \theta_3) = \theta_1 + \theta - 2\theta_3$$

and since $|\theta_1 + \theta_2 - 2\theta_3| < 4 \epsilon$ for any $\epsilon$, it follows that

$$E((x_n - x_m)^2) \to 0$$

hence (Cauchy) $x_n$ tends to a limit.

Conversely, if $x_n \to x$ in the MS sense, then

$$E((x_n - x)^2) \to 0$$

Furthermore,

$$E(x_n^2) = E(x^2)$$

$$E(x_n x_n) = E(x^2)$$

because (see Prob. 6-51)

$$E^2(x_n^2 - x^2) = E^2((x_n - x)(x_n + x))$$

$$\leq E((x_n - x)^2)E((x_n + x)^2) + 0$$

$$E^2(x(x_n - x)) \leq E(x^2)E((x_n - x)^2) \to 0$$

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Similarly, \( E((\bar{x}_n - \bar{x})(\bar{x}_m - \bar{x})) \to 0 \). Hence,

\[
E(\bar{x}_n \bar{x}_m) + E(\bar{x}_m^2) - E(\bar{x}_n \bar{x}_m) - E(\bar{x}_n^2) \to 0
\]

Combining, we conclude that \( E(\bar{x}_n \bar{x}_m) \to E(\bar{x}_m^2) \).

7-27

\[
E(\bar{x}_k) = 0 \quad E(\bar{x}_k^2) = \sigma^2_k
\]

\[
E\left(\left(\sum_{k=n_1}^{n_2} \bar{x}_k\right)^2\right) = \sum_{k=n_1}^{n_2} E(\bar{x}_k^2)
\]

If \( \sum_{k=1}^{\infty} \sigma^2_k < \infty \), then given \( \epsilon > 0 \), we can find \( n_o \) such that \( \sum_{k=n+1}^{n+m} \sigma^2_k < \epsilon \) for any \( m \) and \( n > n_o \). Thus

\[
E((\bar{y}_{n+m} - \bar{y}_n)^2) = E\left(\left(\sum_{k=n+1}^{n+m} \bar{x}_k\right)^2\right) = \sum_{k=n+1}^{n+m} \sigma^2_k < \epsilon
\]

This shows that (Cauchy), \( \bar{y}_k \) converges in the MS sense. The proof of the converse is similar.

7-28

If \( f_i(x) = ce^{-cx} U(x) \) then \( \phi_i(s) = \frac{c}{c-s} \)

\[
\phi(s) = \phi_1(s) \cdots \phi_n(s) = \frac{c^n}{(c-s)^n}
\]

Hence (see Example 5-29) \( f(x) = \frac{e^{nx}x^{-\gamma}}{(n-1)!} e^{-cx} U(x) \)

7-29

From Prob. 7-28 it follows that \( f(x) \) is the density of the sum \( \bar{x} = \bar{x}_1 + \cdots + \bar{x}_n \). Furthermore,

\[
E(\bar{x}) = \frac{n}{c} \quad \sigma^2_x = \frac{n}{c^2}
\]

From the central limit theorem it follows, therefore, that for large \( n \), the Erlang density is nearly equal to a normal curve with mean \( n/c \) and variance \( n/c^2 \).
7-30 \[ E[r_i] = 500 \quad \sigma^2_i = 50^2/3 \]
\[ r = r_1 + r_2 + r_3 + r_4 \quad E[r] = 2000 \quad \sigma^2_r = 10^4/3 \]

Thus, \( r \) is approximately \( N(2000; 10^2/\sqrt{3}) \)

\[ P\{1900 \leq r \leq 2100\} = 2 \cdot G\left(\frac{100\sqrt{3}}{100}\right) - 1 = 0.9169. \]

7-31 The RVs \( x_i \) are independent with (see Prob. 5-37)

\[ f_i(x) = \frac{c_i}{\pi (c_i^2 + x^2)} \quad \Phi_i(\omega) = e^{-c_i|\omega|} \]

In that case, (7-104) does not hold because

\[
\int_{-\infty}^{\infty} x^\alpha f(x) dx = \frac{c_i}{\pi} \int_{-\infty}^{\infty} \frac{x^\alpha}{c_i^2 + x^2} dx = \infty \quad \alpha > 2
\]

In fact, the density of \( \bar{x} = x_1 + \cdots + x_n \) is Cauchy with parameter \( c = c_1 + \cdots + c_n \) because

\[ \Phi(\omega) = e^{-c_1|\omega|} \cdots e^{-c_n|\omega|} = e^{-(c_1 + \cdots + c_n)|\omega|} \]

7-32 In this problem, \( \sigma^2_s = E[|z|^2] = E(x^2 + y^2) = 2\sigma^2 \)

\[ f_s(x) = f_x(x)f_y(y) = \frac{1}{2\pi \sigma^2} e^{-(x^2+y^2)/2\sigma^2} = \frac{1}{2\pi \sigma s^2} e^{-|z|^2/\sigma_s^2} \]

\[ \Phi_s(\Omega) = \Phi_x(u)\Phi_y(v) = \exp \left\{ -\frac{1}{2} \sigma^2(u^2+v^2) \right\} = \exp \left\{ -\frac{1}{4} \sigma_s^2|\Omega|^2 \right\} \]
CHAPTER 8

8-1  (a) From (8-11) with $\gamma=.95$, $u=.975$, $z_{.975} \approx 2$, $\sigma=0.1$, and $n=9$ we obtain

\[ c = \frac{z_{.975} \sigma}{\sqrt{n}} = 0.066 \]

(b) From (8-11) with $c=91.01-91=0.05\text{mm}$:

\[ z_u = \frac{c / \sqrt{n}}{\sigma} = 1.5 \quad u = .933 \quad \gamma = .866 \]

8-2  (a) From (8-11) with $\sigma=1$ and $n=4$: $\bar{x} \pm \sigma z_{.9} / \sqrt{n} \approx 203 \pm 1\text{mm}$

(b) From (8-12) with $\delta=.05$: $c = \sigma / \sqrt{n} \delta = 2.236\text{mm}$

8-3  From (8-4) with $\gamma=.9$, $u=.95$: $\bar{x} \pm z_{.9} \sigma / \sqrt{n} = 25,000 \pm 1,028 \text{ miles}$

8-4  We wish to find $n$ such that $P(|\bar{x} - a| < 0.2) = 0.95$ where $a=E(\bar{x})$. From (8-4) it follows with $u=.975$ and $\sigma=0.1\text{mm}$ that

\[ \frac{z_{.975} \sigma}{\sqrt{n}} \leq 0.2, \text{ hence, } n=1 \]

8-5  In this problem, $x$ is uniform with $E(x) = \theta$ and $\sigma^2=4/3$. We can use, however, the normal approximation for $\bar{x}$ because $n=100$. With $\gamma=.95$, (8-11) yields the interval

\[ \bar{x} \pm z_{.975} \sigma / \sqrt{n} = 30 \pm 0.227 \]
We shall show that if \( f(x) \) is a density with a single maximum and
\[
P(a < x < b) = \gamma, \text{ then } b-a \text{ is minimum if } f(a) = f(b). \text{ The density } xe^{-x}U(x)
\]
is a special case. It suffices to show that \( b-a \) is not minimum if \( f(a) < f(b) \)
or \( f(a) > f(b) \).

Suppose first that \( f(a) < f(b) \) as in figure (a). Clearly, \( f'(a) > 0 \) and \( f'(b) < 0 \), hence, we can find two constants \( \delta_2 > 0 \) and \( \delta_1 > 0 \) such that
\[
P(a + \delta_1 < x < b + \delta_2) = \gamma \quad \text{and} \quad f(a) < f(a + \delta_1) < f(b + \delta_2) < f(b)
\]
From this it follows that \( \delta_1 > \delta_2 \), hence, the length of the new interval
\((a + \delta_1, b + \delta_2)\) is smaller than \( b-a \).

If \( f(a) > f(b) \), we form the interval \((a - \delta_1, b - \delta_2)\) (Fig. 8-6b) and proceed similarly.

Special case. If \( f(x) = xe^{-x} \) then (see Problem 4-9) \( F(x) = 1 - e^{-x} - xe^{-x} \), hence,
P(a < x < b) = e^{-a} + ae^{-a} - e^{-b} - be^{-b} = .95

And since f(a)=f(b), the system

\[ ae^{-a} = be^{-b} \quad e^{-a} - e^{-b} = .95 \]

results. Solving, we obtain a≈0.04  b≈5.75.

A numerically simpler solution results if we set

\[ 0.025 = P(x \leq a) = F(a) \quad 0.025 = P(x > b) = 1 - F(b) \]

as in (9-5). This yields the system

\[ 0.025 = 1 - e^{-a} - ae^{-a} \quad 0.025 = e^{-b} + be^{-b} \]

Solving, we obtain a=0.242, b=5.572. However, the length 5.572-0.242=5.33 of the resulting interval is larger than the length 4.75-0.04=4.71 of the optimum interval.

We start with the general problem: We observe the n samples \( x_i \) of an \( N(\eta,10) \) RV \( x \) and we wish to predict the value \( x \) of \( x \) at a future trial in terms of the average \( \bar{x} \) of the observations. If \( \eta \) is known, we have an ordinary prediction problem. If it is unknown, we must first estimate it. To do so, we form the RV \( w = x - \bar{x} \). This RV is
\[ N(0, \sigma_w) \text{ where } \sigma_w^2 = \sigma_x^2 + \sigma_z^2 - \sigma^2/\eta. \text{ With } c = z_{.975} \sigma_w, \text{ it follows that} \]

\[ P(|\eta| < c) = .95. \text{ Hence} \]

\[ P(\bar{x} - c < \bar{x} < \bar{x} + c) = 0.95 \]

For \( n = 20 \) and \( \sigma = 10 \) the above yields \( \sigma_w = 10.25 \) and \( c \approx 20.5 \). Thus, we can expect with .95 confidence coefficient that our bulb will last at least \( 80 - 20.5 = 59.5 \) and at most \( 80 + 20 = 100.5 \) hours.

\[ \text{8-8} \]

The time of arrival of the 40th patient is the sum \( x_1 + \ldots + x_n \) of \( n = 39 \) RVs with exponential distribution. We shall estimate the mean \( \eta = 1/\theta \) of \( x \) in terms of its sample mean \( \bar{x} = 240/39 = 6.15 \) minutes using two methods. The first is approximate (large \( n \)) and is based on (8-11).

Normal approximation. With \( \lambda = \eta \) and \( z_{.975}/\sqrt{39} = 0.315 \):

\[
P\left( \frac{\bar{x}}{1.315} < \eta < \frac{\bar{x}}{0.683} \right) = .95 \quad 4.68 < \eta < 8.98 \text{ minutes} \]

Exact solution. The RVs \( \bar{x}_i \) are i.i.d. with exponential distribution.

From this and (7-52) it follows that their sum

\[ y = \bar{x}_1 + \ldots + \bar{x}_n = n \bar{x} \text{ has an Erlang distribution:} \]

\[ \text{115} \]
\[ \Phi_y(s) = \frac{\theta^n}{(\theta-s)^n} \quad f_y(y) = \frac{\theta^n}{(n-1)!} y^{n-1} e^{-\theta y} U(y) \]

and the RV \( Z = 2\bar{y} \) has a \( \chi^2(2n) \) distribution:

\[ f_z(z) = \frac{1}{2\theta} f_y \left( \frac{z}{2\theta} \right) U(z) = \frac{z^{n-1}}{2^n(n-1)!} e^{-z/2} U(z) \]

Hence,

\[ P \left\{ \chi^2_{\delta/2}(2n) < \frac{2nY}{\eta} < \chi^2_{1-\delta/2}(2n) \right\} = \gamma = 1-\delta \]

Since \( \chi^2_{.025}(78) = 54.6, \chi^2_{.975}(78) = 104.4 \), and \( 2n\eta = 480 \), this yields the interval

\[ 4.60 < \eta < 8.79 \text{ minutes} \]

\[ \begin{array}{c}
8 - 9 \\
From (8 - 1.9) with \( \bar{x} = 2.550 / 200 = 12.75 \), \( n = 200 \) and \( z_u \approx 2 \)
\end{array} \]

\[ \lambda^2 - 25.52 \lambda + 12.75^2 = 0 \quad \lambda_1 = 12.255 < \lambda < 13.265 = \lambda_2 \]

\[ \begin{array}{c}
8 - 10 \\
From (8 - 21) with \( \bar{x} = 2.080 / 4000 = 0.52 \), \( n = 4000 \) and \( z_u \approx 2.326 \).
\end{array} \]

\[ p_{1,2} = \pi \pm z_u \sqrt{\frac{(\bar{x}(1-\bar{x})}{n}} = .52 \pm .018 \]

Hence, \(.502 < p < .538.\)
8-11 (a) In this problem, \( x = 0.40, n = 900 \) and \( z_u \approx 2 \). From (8-21): Margin of error

\[
\pm 100 z_u \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = \pm 3.27\%
\]

(b) We wish to find \( z_u \). From (9-21) and Table 1a:

\[
100z_u \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = 2 \quad z_u = 1.225 \quad u = .89
\]

This yields the confidence coefficient \( \gamma = 2u - 1 = .78 \)

8-12 From (8-21) with \( \hat{p} = 0.29 \) and \( z_u = 2 \):

\[
z_u \sqrt{\frac{\hat{p}(1-\hat{p})}{n}} = 0.04
\]

\[
n > \frac{n \hat{p}(1-\hat{p})}{0.04^2} \quad z_u^2 = 515
\]

8-13 The problem is to find \( n \) such that [see (8-20)] \( z_u \sqrt{\frac{p(1-p)}{n}} \leq .02 \)

for every \( p \). Since \( z_u \approx 2 \) and \( p(1-p) \leq 1/4 \), this is the case if

\[
z_u \sqrt{1/4n} \leq .02 \quad n \geq 2,500
\]

8-14 From (8-36) with \( k = 1 \)

\[
f(p) = \begin{cases} 
5 & 0.4 < p < 0.6 \\
0 & \text{otherwise}
\end{cases}
\]

\[
P(k=1) = 5 \int_{0.4}^{0.6} p dp = 0.5 = \frac{1}{\gamma}
\]

\[
f_p(p|1) = \begin{cases} 
10p & 0.4 < p < 0.6 \\
0 & \text{otherwise}
\end{cases}
\]

\[
p = 10 \int_{0.4}^{0.6} p^2 dp = 0.5067
\]
From Prob. 8-8: \( f_{\tilde{x}}(x|\theta) = \frac{(\theta n)^n}{(n-1)!} x^{n-1}e^{-n\tilde{x}} \)

From (8-32): \( f_{\tilde{x}}(\theta | x) = \frac{(c+n\tilde{x})^{n+1}}{n!} \theta^n e^{-(c+n\tilde{x})\theta} \)

From (8-31): \( \hat{\theta} = \frac{(c+n\tilde{x})^{n+1}}{n!} \int_0^\infty \theta^{n+1} e^{-(c+n\tilde{x})\theta} d\theta = \frac{n+1}{c+n\tilde{x}} \)

8-16

The sum \( n\tilde{x} \) is a Poisson RV with mean \( n\theta \) (see Prob. 8-8). In the context of Bayesian estimation, this means that

\[ f_{\tilde{x}}(x|\theta) = e^{-n\theta} \frac{(n\theta)^k}{k!} \quad k = n\tilde{x} = 0,1,... \]

Inserting into (8-32), we obtain [see (4-76)]

\[ f_{\tilde{x}}(\theta | x) = \frac{(n+c)^{n\tilde{x}+b+1}}{\Gamma(n\tilde{x}+b+1)} \theta^{n\tilde{x}+b} e^{-(n+c)\theta} \]

and (8-31) yields

\[ \hat{\theta} = \frac{(n+c)^{n\tilde{x}+b+1}}{\Gamma(n\tilde{x}+b+1)} \frac{\Gamma(n\tilde{x}+b+2)}{(n+c)^{n\tilde{x}+b+2}} = \frac{n\tilde{x}+b+1}{n+c} \quad \xrightarrow{n \to \infty} \tilde{x} \]

8-17

From (8-17) with \( t_{.95}(9) = 2.26 \)

\[ x \pm \frac{t_{.05}}{\sqrt{n}} = 90 \pm 3.57 \quad 86.43 < \eta < 93.57 \]

From (8-24) with \( \chi^2_{.975}(9) = 19.02 \), \( \chi^2_{.025}(9) = 2.70 \).
\[
\frac{9 \times 5^2}{19.02} = 11.83 < \sigma^2 < \frac{9 \times 5^2}{2.70} = 83.33 \quad 3.44 < \sigma < 9.13
\]

8-18 The RVs \( x_i / \sigma \) are N(0,1), hence, the sum \( z=(x_1^2 + \cdots + x_{10}^2) / \sigma^2 \) has a \( \chi^2(10) \) distribution. This yields

\[
P(\chi^2_{.025}(10) < z < \chi^2_{.975}(10)) = .95
\]

\[
\chi^2_{.025}(10) = 3.25 < \frac{4}{\sigma^2} < \chi^2_{.975}(10) = 20.48
\]

\[
0.442 < \sigma < 1.109
\]

8-19 From (8-23) with \( n=4, \chi^2_{.025}(4)=0.48, \chi^2_{.975}(4)=11.14 \)

\[
n \hat{\sigma} = .1^2 + .15^2 + .05^2 + .04^2 = .0366
\]

\[
\frac{.0366}{.048} > \sigma^2 > \frac{.0366}{11.14} \quad 0.276 > \sigma > 0.057
\]

8-20 In this problem \( n=3, x_1+x_2+x_3=9.8 \)

\[
f(x,c) \sim c^4 x^3 e^{-cx} \quad f(X,c) = c^4 n(x_1 \cdots x_n) 3n e^{-c\bar{x}}
\]

\[
\frac{\partial f(X,c)}{\partial c} = \left( \frac{4n}{c} - n\bar{x} \right) f(X,c) = 0 \quad \hat{c} = \frac{4}{\bar{x}} = 1.224
\]

8-21 The joint density

\[
f(X,c) = c^n e^{-cn(\bar{x}-x_0)} \quad x_i > x_0
\]

has an interior maximum if

\[
\frac{\partial f(X,c)}{\partial c} = 0 \quad \hat{c} = \frac{1}{\bar{x}-x_0}
\]
The probability

\[ p = 1 - F_x(200) = e^{-200c} \]

of the event \( \{x > 200\} \) is a monoton decreasing function of \( c \). To find the ML estimate \( \hat{c} \) of \( c \) it suffices to find the ML estimate \( \hat{p} \) of \( p \). From Example 8-28 it follows with \( k=62 \) and \( n=80 \) that

\[ \hat{p} = \frac{62}{80} = .775 \text{ hence} \]

\[ \hat{c} = -\frac{1}{200} \ln \hat{p} = 0.0013 \]

The samples of \( x \) are the integers \( x_i \) and the joint density of the RVs \( x_i \) equals

\[ f(X, \theta) = e^{-n\theta} \prod x_i^{x_i} \frac{\theta^{x_i}}{x_i!} = e^{-n\theta} \frac{\theta^{n\bar{x}}}{n\bar{x}!} \]

Hence, \( f(X, \theta) \) is maximum if \( - n + n\bar{x}/\theta = 0 \). This yields \( \hat{\theta} = \bar{x} \)

If \( L = \ln f(x, \theta) \) then

\[ \frac{\partial L}{\partial \theta} = \frac{1}{f} \frac{\partial f}{\partial \theta} \]

\[ \frac{\partial^2 L}{\partial \theta^2} = \frac{1}{f} \frac{\partial^2 f}{\partial \theta^2} - \frac{1}{f} \left( \frac{\partial f}{\partial \theta} \right)^2 \]

\[ \frac{\partial^2 L}{\partial \theta^2} + \left( \frac{\partial L}{\partial \theta} \right)^2 = \frac{1}{f} \frac{\partial^2 f}{\partial \theta^2} \]

But

\[ E \left\{ \frac{1}{f} \frac{\partial f}{\partial \theta} \right\} = \int f \frac{\partial f}{\partial \theta} f dX = 0 \text{ hence} \]

\[ E \left\{ \frac{\partial^2 L}{\partial \theta^2} + \left( \frac{\partial L}{\partial \theta} \right)^2 \right\} = 0 \]
(a) From (8-307): Critical region

\[ x > c = \eta_0 + z_{1-\alpha} \frac{\sigma}{\sqrt{n}} = 8 + 2.326 \times \frac{2}{8} = 8.58 \]

If \( \eta = 8.7 \), then \( \eta_q = \frac{8.7 - 8}{2.18} = 2.8 \)

\[ \beta (\eta) = G(2.36 - 2.8) = .32 \]

(b) We assume that \( \alpha = .01, \beta (8.7) = .05 \) and wish to find \( n \) and \( c \).

\[ G(z_{1-\alpha - \eta_q}) = \beta \quad z_{1-\alpha - \eta_q} = z_\beta \]

\[ \eta_q = z_{.99} - z_{.05} = 4.97 = \frac{8.7 - 8}{2/\sqrt{n}} \]

\[ n = 129 \quad c = 8 + \frac{2}{\sqrt{129}} z_{.99} = 8.41 \]

Our objective is to test the composite null hypothesis \( \eta > \eta_0 = 28 \) against the hypothesis \( \eta < \eta_0 \). Consider first the simple null hypothesis \( \eta = \eta_0 = 28 \). In this case, we can use (8-301) with

\[ q = \frac{x - \eta_0}{s/\sqrt{n}} \quad x = \frac{1}{17} \sum x_i = 27.67 \quad s^2 = \frac{1}{16} \sum (x_i - x) = 17.6 \]

This yields \( s = 4.2 \) and \( q = -0.33 \). Since

\[ q_u = t_u (n-1) = t_{0.05} (16) = -1.95 < -0.33 \]

we conclude that the evidence does not support the rejection of the hypothesis \( \eta = 28 \). The resulting OC function \( \beta_0 (\eta) \) is determined from (9-60c).

If \( \eta_0 > 28 \), then the corresponding value of \( q \) is larger than \(-0.33 \). From this it follows that the evidence does not support the
hypothesis \( \eta_0 \) for any \( \eta_0 > 28 \). We note, however, that the corresponding OC function \( \Phi(\eta) \) is smaller than the function \( \Phi_0(\eta) \) obtained from (8-301) with \( \eta_0 = 28 \).

\[ 8 - 27 \]

From (8-297) with \( q_u = t_u(n-1) \): Critical region \( |x - \eta_0| > t_{1-\alpha/2}(n-1)s/\sqrt{m} \)

1. \( \alpha = .1 \)

\[ t_{.05}(63) = 1.67 \quad |x - \bar{x}| > 1.67 \times 1.5/8 = 0.313 \]

Since \( \bar{x} = 7.7 \) is in the interval \( 8 \pm 0.317 \), we accept \( H_0 \)

2. \( \alpha = .01 \)

\[ t_{.01}(63) = 2.62 \quad |x - \bar{x}| > 2.62 \times 1.5/8 = 0.49 \]

Since \( \bar{x} = 7.7 \) is outside the interval \( 8 \pm 0.49 \), we reject \( H_0 \).

\[ 8 - 28 \]

We assume that the RVs \( x \) and \( y \) are normal and independent. We form the difference \( w = \bar{x} - \bar{y} \) of their sample means

\[ \bar{x} = \frac{1}{16} \sum_{i=1}^{16} x_i \quad \bar{y} = \frac{1}{26} \sum_{i=1}^{26} y_i \]

and use as test statistic the ratio

\[ q = \frac{w}{\sigma_w} \quad \sigma_w^2 = \frac{\sigma_x^2}{16} + \frac{\sigma_y^2}{26} \]

The RV \( q \) is normal with \( \sigma_q = 1 \) and under hypothesis \( H_0 \), \( E(q) = 0 \). We can,
therefore, use (8-307) because \( q_u = z_u \). To find \( q \), we must determine \( \sigma_w \).

Since \( \sigma_x \) and \( \sigma_y \) are not specified, we shall use the approximations \( \sigma_x \approx s_x = 1.1 \) and \( \sigma_y \approx s_y = 0.9 \). This yields

\[
\sigma_w^2 \approx \frac{1.1^2}{16} + \frac{0.9^2}{26} = 0.107 \quad q = \frac{x-y}{\sigma_w} = \frac{0.4}{0.327} = 1.223
\]

Since \( z_{0.95} = 1.645 > 1.223 \), we accept \( H_0 \).

8-29

(a) In this problem, \( n = 64 \), \( k = 22 \), \( p_0 = q_0 = 0.5 \)

\[ q = \frac{k - np_0}{\sqrt{np_0(1-p_0)}} = 2.5 \quad z_{\alpha/2} = - z_{1-\alpha/2} \approx -2 \]

Since 2.5 is outside the interval (2, -2), we reject the fair coin hypothesis [see (8-313)].

(b) From (8-313) with \( n = 16 \), \( p_0 = q_0 = 0.5 \):

\[
\frac{k_1 - np_0}{\sqrt{np_0(1-p_0)}} = z_{\alpha/2} \quad \frac{k_2 - np_0}{\sqrt{np_0(1-p_0)}} = - z_{\alpha/2}
\]

This yields \( k_1 = 8 - 2 \times 2 = 4 \), \( k_2 = 8 + 2 \times 2 = 12 \)

8-30 We shall use as test statistic the sum

\[ q = \sum_{i=1}^{m} x_i \quad n = 22 \]
The critical region of the test is \( q < q_\alpha \) where \( q = x_1 + \cdots + x_n = 90 \) [see (8-301)].

The RV \( q \) is Poisson distributed with parameter \( n\lambda \). Under hypothesis \( H_0 \), \( \lambda = \lambda_0 = 5 \); hence, \( \eta = n\lambda_0 = 110 = \sigma_q^2 \). To find \( q_\alpha \) we shall use the normal approximation. With \( \alpha = 0.05 \) this yields

\[
q_\alpha = n\lambda_0 + z_\alpha \sqrt{n\lambda_0} = 90 - 17.25 = 72.75
\]

Since \( 90 > 72.75 \), we accept the hypothesis that \( \lambda = 5 \).

8-31

From (9-75) with \( n = 102 \) and \( p_{0i} = 1/6 \)

\[
q = \sum_{i=1}^{6} \frac{(k_i - 17)^2}{17} = 2 \quad \chi^2_{0.05}(5) \sim 11
\]

Since \( 2 < 11 \), we accept the fair die hypothesis.

8-32

Uniformly distributed integers from 0 to 9 means that they have the same probability of appearing. With \( m = 10 \), \( p_{01} = 1 \), and \( n = 1,000 \), it follows from (8-325) that

\[
q = \sum_{j=0}^{9} \frac{(n_j - 100)^2}{100} = 17.76 \quad \chi^2_{0.05}(9) = 16.92
\]

Since \( 17.76 > 16.92 \), we reject the uniformity hypothesis.

8-33

In this problem

\[
f(x, \theta) = e^{-\theta} \frac{\theta^x}{x!} \quad f(X, \theta) = \frac{e^{-n\theta} \theta^{nx}}{x_1! \cdots x_n!}
\]

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(X, \theta) \) is maximum for \( \theta = \theta_m = \bar{x} \). And since \( \theta_0 = \theta_0 \) we conclude that

\[
\lambda(X) = \frac{e^{-n\theta_0} \theta^n X^n \bar{x}^n x^n}{e^{-n\bar{x}^n \theta_0^n x^n}} \quad w = -2 \ln \lambda = 2n(\theta_0 - \bar{x}) + \bar{x} \ln(\bar{x}/\theta_0)
\]

With \( n=50, \theta_0=20, \bar{x}=1.058/50=21.16 \), this yields \( w=3 \). Since \( m_0=1, m=1 \), and \( \chi^2_{0.05}(1)=3.84>3 \), we accept \( H_0 \).

8-34 We form the RVs

\[
z = \sum_{i=1}^{m} \left( \frac{X_i - \eta_x}{\sigma_x} \right)^2 \quad w = \sum_{i=1}^{n} \left( \frac{Y_i - \eta_y}{\sigma_y} \right)^2
\]

These RVs are \( \chi^2(m) \) and \( \chi^2(n) \) respectively. If \( \sigma_x = \sigma_y \), then

\[
q = \frac{z/m}{w/n}
\]

Hence (see Prob. 6-23), \( q \) has a Snedecor distribution. To test the hypothesis \( \sigma_x = \sigma_y \), we use (8-297) where \( q_u = F_u(m,n) \) is the tabulated \( u \) percentile of the Snedecor distribution. This yields the following test:

Accept \( H_0 \) iff \( F_{\alpha/2}(m,n) < \alpha < F_{1-\alpha/2}(m,n) \).

8-35 If \( x \) has a student-t distribution, then \( f(-x) = f(x) \), hence (see Prob. 6-75)

\[
E(x) = 0 \quad \sigma_x^2 = E(x^2) = \frac{n}{n-2}
\]

8-36 (a) Suppose that the probability \( P(A) \) that player A wins a set equals \( p=1-q \). He wins the match in five sets if he wins two of the first four sets and the fifth set. Hence, the probability \( p_5(A) \) that he wins in five equals \( 6p^3q^2 \). Similarly, the probability \( p_5(B) \) that player B wins in five equals \( 6p^2q^3 \). Hence,

\[
p_5 = p_5(A) + p_5(B) = 6p^3q^2 + 6p^2q^3 = 6p^2q^2
\]

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is the probability that the match lasts five sets. If $p=q=\frac{1}{2}$, then $p_5=\frac{3}{8}$.

(b) Suppose now that $P(A) = p$ is an RV with density $f(p)$. In this case,

$$p_5 = 6p^2(1-p^2)$$

is an RV. We wish to find its best bayesian estimate. Using the MS criterion, we obtain

$$\hat{p}_5 = E(p_5) = \int_0^1 6p^2(1-p^2)f(p)dp$$

If $f(p)=1$, then $\hat{p}_5 = \frac{1}{5}$.

---

8-37 Given

$$f_\nu(v) \sim e^{-v^2/2\sigma^2} \quad f_\varphi(\theta) \sim e^{-(\theta-\theta_0)^2/2\sigma_0^2}$$

To show that

$$f_\varphi(\theta|x) \sim e^{-(\theta-\theta_1)^2/2\sigma_1^2}$$

where

$$\frac{1}{\sigma_1^2} = \frac{1}{\sigma_\varphi^2} + \frac{n}{\sigma^2} \quad \theta_1 = \frac{\sigma_1^2}{\sigma_\varphi^2} \theta_0 + \frac{n\sigma_1^2}{\sigma^2} \bar{x}$$

Proof

$$f(x|x|\theta) = f_\nu(x-\theta) \sim \exp \left\{ -\frac{(x-\theta)^2}{2\sigma^2} \right\}$$

$$f(X)\theta \sim \exp \left\{ -\frac{1}{2\sigma^2} \sum (x_i-\theta)^2 \right\}$$

Since $\sum (x_i-\theta)^2 = \sum (x_i-x)^2 + n(\theta-\theta)^2$, we conclude from (8-32) omitting factors that do not depend on $\theta$ that

$$f(\theta|X) \sim \exp \left\{ -\frac{1}{2} \left[ \frac{(\theta-\theta_0)^2}{\sigma_\varphi^2} + \frac{n(\bar{x}-\theta)^2}{\sigma^2} \right] \right\}$$

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The above bracket equals
\[
\left( \frac{1}{\sigma_{0}^{2}} + \frac{n}{\sigma^{2}} \right) \theta^{2} - 2 \left( \frac{\theta_{0}}{\sigma_{0}^{2}} + \frac{n\bar{x}}{\sigma^{2}} \right) \theta + \cdots = \frac{1}{\sigma^{2}} \left( \theta^{2} - 2 \theta_{1} \right) + \cdots
\]

and (i) follows.

8-38 The likelihood function of \( X \) equals
\[
f(X, \theta) = \frac{1}{(\sqrt{2\pi\theta})^{n}} \exp \left\{ -\frac{1}{2\theta} \sum (x_{i}-\eta)^{2} \right\}
\]

where \( \theta = \sigma^{2} \) is the unknown parameter. Hence
\[
L(X, \theta) = -\frac{n}{2} \ln (2\pi\theta) + \frac{1}{2\theta} \sum (x_{i}-\eta)^{2}
\]

\[
\frac{\partial L(X, \theta)}{\partial \theta} = -\frac{n}{2\theta} \sum (x_{i}-\theta)^{2} = 0 \quad \hat{\theta} = \frac{1}{n} \sum (x_{i}-\eta)^{2}
\]

8-39 The estimators \( \hat{\theta}_{1} \) and \( \hat{\theta}_{2} \) have the same variance because otherwise one or the other would not be best. Thus

\[
E(\hat{\theta}_{1}) = E(\hat{\theta}_{2}) = \theta \quad \text{var} \hat{\theta}_{1} = \text{var} \hat{\theta}_{2} = \sigma^{2}
\]

If \( \hat{\theta} = \frac{1}{2} (\hat{\theta}_{1} + \hat{\theta}_{2}) \), then

\[
E(\hat{\theta}) = \theta \quad \sigma_{\hat{\theta}}^{2} = \frac{1}{2} \left( \sigma^{2} + \sigma^{2} + 2\rho\sigma^{2} \right) = \frac{1}{2} (1+\rho)\sigma^{2}
\]

where \( \sigma \) is the correlation coefficient of \( \hat{\theta}_{1} \) and \( \hat{\theta}_{2} \). If \( r < 1 \) then \( \sigma_{\hat{\theta}} < \sigma \) which is impossible. Hence, \( r = 1 \) and \( \hat{\theta}_{1} = \hat{\theta}_{2} \) (see Prob. 6-53).

8-40 \( k_{1} + k_{2} - np_{1} - np_{2} = n - n(p_{1} + p_{2}) = 0 \); Hence, \( |k_{1} - np_{1}| = |k_{2} - np_{2}| \)

\[
\frac{(k_{1} - np_{1})^{2}}{np_{1}} + \frac{(k_{2} - np_{2})^{2}}{np_{2}} = (k_{1} - np_{1})^{2} \left( \frac{1}{np_{1}} + \frac{1}{np_{2}} \right) = \frac{(k_{1} - np_{1})^{2}}{np_{1}p_{2}}
\]

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It is given that
\[ E\{T(X)\} = \int_{-\infty}^{\infty} T(X) f(X; \theta) \, dx = \psi(\theta), \]
so that after differentiating and making use of (8.81) we get
\[ \int_{-\infty}^{\infty} T(X) \frac{\partial f(X; \theta)}{\partial \theta} \, dx = \psi'(\theta) \quad (8.41 - 1) \]
Also using (8.80)
\[ \int_{-\infty}^{\infty} \psi(\theta) \frac{\partial f(X; \theta)}{\partial \theta} \, dx = 0, \quad (8.41 - 2) \]
and the above two expressions give
\[ \int_{-\infty}^{\infty} [T(X) - \psi(\theta)] \frac{\partial f(X; \theta)}{\partial \theta} \, dx = \psi'(\theta) \quad (8.41 - 3) \]
But
\[ \frac{\partial f(X; \theta)}{\partial \theta} = \frac{1}{f(X; \theta)} \frac{\partial \log f(X; \theta)}{\partial \theta} \]
so that (8.41-3) simplifies to
\[ \int_{-\infty}^{\infty} \left[ \{T(X) - \psi(\theta)\} \sqrt{f(X; \theta)} \right] \left[ \sqrt{f(X; \theta)} \frac{\partial \log f(X; \theta)}{\partial \theta} \right] \, dx = \psi'(\theta) \]
and application of Cauchy-Schwarz inequality as in (8.89)-(8.92), Text gives
\[ E \left[ \{T(X) - \psi(\theta)\}^2 \right] \geq \frac{[\psi'(\theta)]^2}{E \left\{ \left( \frac{\partial \log f(X; \theta)}{\partial \theta} \right)^2 \right\}} \]
9-1

\( x(t) = e^{at} \)

\[ n(t) = \int_{-\infty}^{\infty} e^{at} f_a(a) \, da \]

\[ R(t_1, t_2) = \int_{-\infty}^{\infty} e^{at_1} e^{at_2} f_a(a) \, da \]

From (5-16) with \( x = g(a) = e^{ta} \)

\( g'(a) = t e^{ta} = tx \)

\[ f(x, t) = \frac{1}{x|t|} f_a \left( \frac{1}{t} \ln x \right) U(x) \]
9-3 As we know, \( \bar{E}(x(t)) = \lambda t \) and \( \text{var} \ x(t) = \lambda^2 t^2 \) [see (9-18)]. But \( \bar{E}(x(9)) = 6 \) by assumption, hence, \( \lambda = 2/3 \)

(a) \( \bar{E}(x(8)) = 24 \quad \text{var} \ x^2(t) = 24^2 \)

(b) TheRV \( x(2) \) is Poisson distributed with parameter \( 2\lambda = 6 \). Hence,

\[
P(x(2) \leq 3) = e^{-2\lambda} \sum_{k=0}^{3} \frac{(2\lambda)^k}{k!}
\]

(c) The RVs \( \bar{z} = x(2) \) and \( \bar{w} = x(4) - x(2) \) are independent and Poisson distributed with parameter \( 2\lambda \). Hence,

\[
P(z=k) = e^{-2\lambda} \frac{(2\lambda)^k}{k!} \quad P(z = k, w = m) = e^{-4\lambda} \frac{(2\lambda)^k (2\lambda)^m}{k! m!}
\]

\[
P(x(4) \leq 5 \mid x(2) \leq 3) = \frac{P(z \leq 5 \mid x(2) \leq 3)}{P(z \leq 3)} \quad P(z \leq 3) = \sum_{k=0}^{3} P(z=k)
\]

\[
P(z \leq 3, w \leq 5 - z) = \sum_{k=0}^{3} \sum_{m=0}^{5-k} P(z = k, w = m)
\]

9-4 \( \chi(t) = U(t - \xi) \quad \chi(t) = \delta(t - \xi) = \chi'(t) \)

For \( t_1 \) or \( t_2 < 0 \), \( R(t_1, t_2) = 0 \); for \( t_1 \) and \( t_2 > T \), \( R(t_1, t_2) = 1 \). Otherwise,

\[
R_x(t_1, t_2) = \frac{1}{T} \min(t_1, t_2) \quad \frac{3R_x}{\partial t_1} = \frac{1}{T} U(t_1 - t_2) \quad - \frac{3R_x}{\partial t_1 \partial t_2} = \frac{1}{T} \delta(t_1 - t_2)
\]

From this and (9-105) it follows that \( TR_y(t_1 - t_2) = \delta(t_1 - t_2) \) for \( 0 < t_1, t_2 < T \) and 0 otherwise.

9-5 \( a - b \ t = 0 \iff t = \xi_1 = a/b \). Setting \( \sigma_1 = \sigma_2 = \sigma \) and \( r = 0 \) in (6-63), we obtain

\[
P(0 < \xi_1 < T) = \frac{1}{2} + \frac{1}{\pi} \arctan T - (\frac{1}{2} + \frac{1}{\pi} \arctan 0)
\]
The equations
\[ y''(t) = y(t)U(t) \]
\[ y(0) = y'(0) = 0 \]

specify a system with input \( y(t)U(t) \) and impulse response \( h(t) = tU(t) \).
Hence [see (9-100)]
\[ E[y^2(t)] = q(t)U(t) \ast t^2U(t) = \int_0^t (t - \tau)q(\tau)d\tau \]

9-7
(a) From (5-88) with \( \xi = x(t + \tau) - x(t) \), and (8-101):
\[ P\{|x(t+\tau) - x(t)| \geq a\} \leq \frac{E[(\xi(t+\tau) - \xi(t))^2]}{a^2} \]
\[ = \frac{2[R(0) - R(\tau)]}{a^2} \]
The above probability equals the mass in the regions (shaded)
\( x_2 - x_1 > a \) and \( x_2 - x_1 < -a \)
Hence,
\[ P\{|x(t+\tau) - x(t)| \geq a\} \]
\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{x_2-a} f(x_1, x_2; \tau)dx_1dx_2 + \int_{-\infty}^{\infty} \int_{x_2+a}^{\infty} f(x_1, x_2; \tau)dx_1dx_2 \]
9-8 (a) The RV \( x(t) \) is normal with zero mean and variance \( \bar{E}(x^2(t)) = R(0) = 4 \), hence it is \( N(0,2) \) and \( \bar{P}(x(t) \leq 3) = F(3) = G(1.5) = 0.933 \)

(b) \( \bar{E}[(x(t+1) - x(t-1))] = 2[R(0)-R(2)] = 8(1-e^{-4}) \)

9-9 If \( x(t) = ce^{j(\omega t+\theta)} \) and \( \eta_c = 0 \) then

\[
\eta_x(t) = \eta_c e^{j(\omega t+\theta)} = 0 \quad \Rightarrow \quad R_{xx}(t+r,t) = \sigma_c^2 e^{j\omega r}
\]

hence, \( x(t) \) is WSS. We shall prove the converse:

If the process \( x(t) = c w(t) \) is WSS, then \( \eta_c = 0 \) and \( w(t) = e^{j(\omega t+\theta)} \) within a constant factor.

Proof \( \eta_x(t) = \eta_c w(t) \) is independent of \( t \); hence, \( \eta_c = 0 \). The function

\[
R_{xx}(t_1,t_2) = \sigma_c^2 w(t_1)w^*(t_2) \text{ depends only on } r = t_1 - t_2; \text{ hence, } w(t+r)w^*(t) = g(r). \text{ With } r=0 \text{ this yields}
\]

\[
|w(t)|^2 = g(0) = \text{constant} \quad \Rightarrow \quad w(t) = ae^{j\phi(t)}
\]

\[
w(t+r)w^*(t) = a^2 e^{j[\phi(t+r)-\phi(t)]}
\]

Hence the difference \( \phi(t+r)-\phi(t) \) depends only on \( r \):

\[
\phi(t+r)-\phi(t) = f(r) \quad (i)
\]

From this it follows that, if \( \phi(t) \) is continuous then, \( \phi(t) \) is a linear function of \( t \). To simplify the proof, we shall assume that \( \phi(t) \) is differentiable. Differentiating with respect to \( t \), we obtain \( \phi'(t+r) = \phi'(t) \) for every \( r \). With \( r=0 \) this yields

\[
\phi'(r) = \phi'(0) = \text{constant} \quad \Rightarrow \quad \phi(r) = ar+b
\]

9-10 We shall show that if \( x(t) \) is a normal process with zero mean and \( z(t) = x^2(t) \), then \( C_{zz}(r) = 2C_{xx}z^2(r) \).

From (7-61): If the RVs \( x_k \) are normal and \( E(x_k) = 0 \), then
\[ E(x_1 x_2 x_3 x_4) = E(x_1 x_2) E(x_3 x_4) + E(x_1 x_3) E(x_2 x_4) + E(x_1 x_4) E(x_2 x_3) \]

With \( x_1 = x_2 = x(t+r) \) and \( x_3 = x_4 = x(t) \), we conclude that the autocorrelation of \( x(t) \) equals

\[ E(x^2(t+r)x^2(t)) = E^2(x^2(t+r)) + 2E^2(x(t+r)x(t)) = R_{xx}(0) + 2R_{xx}(r) \]

And since \( R_{xx}(r) = C_{xx}(r) \), and \( E(z(t)) = R_{xx}(0) \), the above yields

\[ C_{xx}(r) = R_{xx}(r) - E^2(z(t)) = 2C_{xx}(r) \]

9-11

\[ y''(t) + 4y'(t) + 13y(t) = x(t) \text{ all } t \]

The process \( y(t) \) is the response of a system with input \( x(t) = 26 + \nu(t) \) and

\[ H(s) = \frac{1}{s^2 + 45 + 13} \quad h(t) = \frac{1}{3} e^{-2t} \sin 3t U(t) \]

Since \( n_x = 26 \), this yields \( n_y = n_x H(0) = 2 \). The centered process \( \bar{y}(t) = y(t) - n_y \) is the response due to \( \nu(t) \). Hence [see (9-100)]

\[ E(\bar{y}^2(t)) = q \int_0^\infty h^2(t) dt = \frac{10}{104} \]

With \( b=4 \) and \( c=13 \) it follows that (see Example 9-276)

\[ R_{y^2}(r) = \frac{10}{104} e^{-2|r|} \left( \cos 3r - \frac{2}{3} \sin 3|r| \right) + 4 \]

If \( \nu \) is normal, then \( y(t) \) is normal with mean 2 and variance \( R_{yy}(0) - 4 = 10/104 \); hence,

\[ P(y(t) \leq 3) = G \left( \frac{3 - 2}{0.31} \right) = G(3.24) \]

9-12

\[ E(y(t)) = 0 \quad R_{yy}(t_1, t_2) = \frac{R_{xx}(t_1, t_2)}{f(t_1) f(t_2)} = w(t_1 - t_2) \]

\[ E(z(t)) = 0 \quad R_{zz}(t_1, t_2) = \frac{R_{xx}(t_1, t_2)}{\sqrt{q(t_1)} \sqrt{q(t_2)}} = \delta(t_1 - t_2) \]

because \( q(t_1) \delta(t_1 - t_2) = \sqrt{q(t_1)} \sqrt{q(t_2)} \delta(t_1 - t_2) \).
From (9-181) and the identity \(4ab \leq (a+b)^2\) it follows that
\[
| R_{xy}(\tau) |^2 \leq R_{xx}(0)R_{yy}(0) \leq \frac{1}{4} [ R_{xx}(0) + R_{yy}(0) ]^2
\]

9-14  Clearly (stationarity assumption)
\[
E[| x^*(t) - y^*(t) |^2] = E[| x(0) - y(0) |^2] = 0
\]
Furthermore,
\[
E\{x(t+\tau)[x^*(t) - y^*(t)]\} = R_{xx}(\tau) - R_{xy}(\tau)
\]
and [see (9-177)]
\[
|E\{x(t+\tau)[x^*(t) - y^*(t)]\}|^2 \leq E(|x(t+\tau)|^2)E(|x^*(t) - y^*(t)|^2) = 0
\]
Hence, \(R_{xx}(\tau) - R_{xy}(\tau) = 0\); similarly, \(R_{yy}(\tau) = R_{xy}(\tau)\)

9-15  \[
E[| x(t+\tau) - x(t) |^2] = E\{[x(t+\tau) - x(t)][x^*(t+\tau) - x^*(t)]\}
= R(0) - R(\tau) - R^*(\tau) + R(0) = 2R(0) - 2 \text{ Re } R(\tau)
\]

9-16  From \(\Phi(1) = \Phi(2) = 0\) it follows that
\[
E\{\cos \phi\} = E\{\sin \phi\} = E\{\cos 2\phi\} = E\{\sin 2\phi\} = 0
\]
Hence, \(E\{\chi(t)\} = \cos \omega t E\{\cos \phi\} - \sin \omega t E\{\sin \phi\} = 0\)
and as in Example 9-14
\[
2 \cos [\omega(t+\tau) + \phi] \cos(\omega t + \phi) = \cos \omega \tau + \cos(2\omega t + \omega \tau + 2\phi)
\]
\[
2 R_x(\tau) = \cos \omega \tau
\]
If \(\phi\) is uniform in \((-\pi, \pi)\), then
\[
\Phi(\lambda) = \frac{\sin \pi \omega}{\pi \omega} \quad \Phi(1) = \Phi(2) = 0
\]
9-17 (a) \[ \dot{x}(t_1)x(t_2) = [\dot{x}(t_1) - \dot{x}(0)][x(t_2) - x(t_1) + x(t_1) - x(0)] \]

\[ R(t_1, t_2) = E[(\dot{x}(t_1) - \dot{x}(0))^2] = E(\dot{x}^2(t_1)) = R(t_1, t_1) \]

(b) If \( t_1 + \epsilon < t_2 \), then \( R_y(t_1, t_2) = 0 \); if \( t_1 < t_2 < t_1 + \epsilon \) then

\[ E([\dot{x}(t_1 + \epsilon) - \dot{x}(t_1)][\dot{x}(t_2 + \epsilon) - \dot{x}(t_2)]) = q(t_1 + \epsilon - t_2) \]

Hence, \( \epsilon^2 R_y(\tau) = q(\epsilon - |\tau|) \) for \( |\tau| = |t_2 - t_1| < \epsilon \)

9-18

\[ E(\ddot{x}(t)\ddot{x}(t)) = \int_{-\infty}^{\infty} E(\ddot{x}(t)\ddot{x}(t - \tau))h(\tau)d\tau \]

\[ = \int_{-\infty}^{\infty} R_{xx}(t, t - \tau)h(\tau)d\tau = \int_{-\infty}^{\infty} q(\tau)\delta(\tau)h(\tau)d\tau = h(0)q(t) \]

9-19 As in Prob. 5-14, \( g(x) = 6 + 3 F_x(x) \). In this case,

\[ E(\dot{x}^2(t)) = 4, \text{ hence, } \dot{x}(t) \text{ is } N(0, 2) \text{ and } F_x(x) = G(x/2) \]

9-20 \( \dot{x}(t) \) is SSS, hence, \( P(x(t) < y) = F_x(y) \) does not depend on \( t \). The RVs \( \xi \) and \( \ddot{x}(t) \) are independent, hence, [see (6-238)]

\[ F_y(y) = P(\dot{x}(t - \epsilon) < y | \xi = \epsilon) = P(\dot{x}(t - \epsilon) < y | \dot{x} = \epsilon) \]

\[ = P(\dot{x}(t - \epsilon) < y) = F_x(y) \]

is independent of \( t \). Similarly for higher order distributions.
9-21 \( E(\chi(t)) = \eta = \text{constant, hence, [see (9-102)] } E(\chi'(t)) = 0 \)

Furthermore, \( R_{xx}(-\tau) = R_{xx}(\tau) \). hence, \( R_{xx}'(0) = 0 \) and (10-97) yields

\[
E(\chi(t)\chi'(t)) = R_{xx}(0) = 0
\]

9-22 (a) \( E(z) = R_x(2) = 4e^{-4} \quad E(z^2) = E(w^2) = R_x(0) = 4 \)

\[
E((z+w)^2) = R_x(0) + R_x(0) + 2R_x(2) = 8(1+e^{-4})
\]

(b) \( z \sim N(0,2) \quad P(z < 1) = F_z(1) = G(1/2) \)

\[
r_{zw} = e^{-4}, \quad f_{zw}(z,w) : N(0,0;2,2;e^{-4})
\]

9-23 The RV \( \chi'(t) \) is normal with zero mean and variance

\[
E(\chi'(t))^2 = R_{\chi',\chi'}(0) = -R''(0)
\]

Hence, \( P(\chi'(t) < a) = F_{\chi',\chi'}(a) = G[a/\sqrt{R''(0)}] \)

9-24 The function \( \arcsin x \) is odd, hence, it can be expanded into a sine series in the interval \((-1,1)\):

\[
\arcsin x = \sum_{n=1}^{\infty} b_n \sin n\pi x \quad |x| \leq 1
\]

\[
b_n = \frac{1}{\pi} \int_{-1}^{1} \arcsin x \sin n\pi x \, dx = \frac{1}{n\pi} \int_{-1}^{1} \arcsin x \cos n\pi x \, dx
\]

\[
= -\frac{\arcsin x}{n\pi} \left| \left. \frac{\cos n\pi x}{n\pi} \right|_{-1}^{1} + \frac{1}{n\pi} \int_{-1}^{1} \cos n\pi x \, dx
\]

\[
= -\frac{\cos n\pi}{n\pi} + \frac{1}{n\pi} \int_{-\pi/2}^{\pi/2} \cos(n\pi \sin x) \, dx
\]

and the result follows because [see (9-81)]

\[
R_y(\tau) = 2 \pi \frac{\arcsin \frac{R_x(\tau)}{R_x(0)}}{J_0(z) = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} \cos(z \sin x) \, dx}
\]
As we know [see (5-100) and (6-193)]

\[ E[e^{j\omega x(t)}] = \exp\left\{ -\frac{R(0)}{2} \omega^2 \right\} \]

\[ E\{e^{j[\omega_1 x(t+\tau)+\omega_2 x(t)]} \} = \exp \left\{ -\frac{1}{2} \left[ R(0)\omega_1^2 + 2R(\tau)\omega_1\omega_2 + R(0)\omega_2^2 \right]\right\} \]

Hence, with \( j\omega = a \)

\[ E\{I e^{ax(t)}\} = \exp \left\{ \frac{1}{2} R_x(0) \right\} I \]

\[ E\{I e^{ax(t+\tau)}I e^{ax(t)}\} = I^2 \exp \{ a [R_x(0) + R_x(\tau)]\} \]

(a) \( R_y(\tau) = a^2 E[x(c(t+\tau))x(ct)] = a^2 R(c\tau) \)

(b) If \( z_\varepsilon(t) = \sqrt{\varepsilon} x(\varepsilon t) \) then \( R_{z_\varepsilon}(\tau) = \varepsilon R_x(\varepsilon \tau) \) [as in (a)].

If \( \delta > 0 \) is sufficiently small and \( \phi(t) \) is continuous at the origin, then

\[ \int_{-\delta}^{\delta} R_{z_\varepsilon}(\tau)\phi(\tau)d\tau = \phi(0) \int_{-\delta}^{\delta} \varepsilon R_x(\varepsilon \tau)d\tau \]

\[ = \phi(0) \int_{-\varepsilon\delta}^{\varepsilon\delta} R(\tau)d\tau \overset{\varepsilon \rightarrow \infty}{\longrightarrow} \phi(0) \int_{-\infty}^{\infty} R(\tau)d\tau = q \phi(0) \]

Hence, \( R_{z_\varepsilon}(\tau) \rightarrow q \delta(\tau) \) as \( \varepsilon \rightarrow \infty \).
\[ y(t) = \int_{t-T}^{t} \chi(\tau)h(t-\tau)d\tau \]

Hence, \( y(t_1) \) and \( y(t_2) \) depend linearly on the values of \( \chi(t) \) in the intervals \((t_1 - T, t_1)\) and \((t_2 - T, t_2)\) respectively. If \(|t_1 - t_2| > T\) then these intervals do not overlap and since 
\[ E\{\chi(t_1)\chi(t_2)\} = 0 \quad \text{for} \quad t_1 \neq t_2, \]
it follows that 
\[ E\{y(t_1)y(t_2)\} = 0. \]

\[ I(t) = E\left\{ \sum_{\alpha,\beta} h(t,\alpha)\chi(\alpha)h(t,\beta)\chi(\beta) \right\} \]

\[ = \int_{0}^{t} \int_{0}^{t} h(t,\alpha)h(t,\alpha)q(\alpha)\delta(\alpha-\beta)d\alpha d\beta = \int_{0}^{t} h^2(t,\alpha)q(\alpha)d\alpha \]

(a)

(b) If \( \chi'(t) + c(t)\chi(t) = \chi(t) \), then \( \chi(t) \) is the output of a linear time-varying system as in (a) with impulse response \( h(t,\alpha) \) such that
\[ \frac{\partial h(t,\alpha)}{\partial t} + c(t)h(t,\alpha) = \delta(t-\alpha) \quad \text{and} \quad h(\alpha^-,\alpha) = 0 \]
or equivalently
\[ \frac{\partial h(t,\alpha)}{\partial t} + c(t)h(t,\alpha) = 0 \quad \text{if} \quad t > 0 \quad h(\alpha^+,\alpha) = 1 \]

This yields
\[ h(t,\alpha) = e^{-\int_{0}^{\alpha} c(\tau)d\tau} \]

Hence, if
\[ I(t) = \int_{0}^{t} h^2(t,\alpha)q(\alpha)d\alpha \]

then \( I'(t) + 2c(t)I(t) = q(t) \)

because the impulse response of this equation equals
\[ e^{-\int_{0}^{t} c(\tau)d\tau} \quad = h^2(t,\alpha) \]
9-29

(a) If \( y'(t) + 2y(t) = x(t) \), then \( y(t) = x(t) * h(t) \)

where \( h(t) = e^{-2t}u(t) \) and with \( q(t) = 5 \), (10-90) yields

\[
E[y^2(t)] = 5 \int_0^\infty e^{-4\tau}u(t) \, d\tau = \frac{5}{4}
\]

(b) As in (a) with \( q(t) = 5u(t) \). Hence, for \( t > 0 \)

\[
E[y^2(t)] = 5u(t)e^{-4t}u(t) = 5 \int_0^t e^{-4\tau} \, d\tau = \frac{5}{4} (1 - e^{-4t})
\]

From (9-90) with \( q(t) = N[u(t) - u(t - T)] \)

\[
E[y^2(t)] = \begin{cases} 
AN \int_0^t e^{-2a(t-\tau)} \, d\tau = \frac{AN}{2a} (1 - e^{-2at}) & 0 \leq t < T \\
AN \int_T^t e^{-2a(t-\tau)} \, d\tau = \frac{AN}{2a} (e^{2aT} - 1)e^{-2at} & t > T 
\end{cases}
\]

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9-31 Since \( x(t) \) is WSS, the moments of \( S \) equal the moments of

\[
Z = \int_{-\infty}^{\infty} x(t) \, dt
\]

Hence, (see Fig. 9-5)

\[
E[s^2] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_x(t_1, t_2) \, dt_1 \, dt_2 \int_{-10}^{10} \int_{-10}^{10} (10-|\tau|) R_x(\tau) \, d\tau
\]

\[
E[s] = 80 \quad \sigma_s^2 = 2 \int_{0}^{10} (10-\tau) 10e^{-2\tau} \, d\tau
\]

9-32 \( y(t) = x(t) \ast h(t) \)

\[
h(t) = e^{-2t} U(t)
\]

(a) \( E[y^2(t)] = 5e^{-4t} \) \( U(t) = 5/4 \)

\[
R_{xy}(t_1, t_2) = 5 \delta(t_1-t_2) e^{-2t_2} U(t_2) = 5e^{-2(t_2-t_1)} U(t_2-t_1)
\]

\[
R_{yy}(t_1, t_2) = 5e^{-2(t_2-t_1)} U(t_2-t_1) e^{-t_1} U(t_1)
\]

\[
= \frac{5}{4} e^{-2|t_1-t_2|}
\]

The first equation follows from (9-100) with \( q(t) = 5 \); the second from (9-94) with \( R_{xx}(t_1, t_2) = 5\delta(t_1-t_2) \), and the third from (9-96).

(b) With \( R_{xx}(t_1, t_2) = 5\delta(t_1-t_2) U(t_1) U(t_2) \), (9-94) and (9-96) yield the following: For \( t_1 \) or \( t_2 < 0 \), \( R_{xy}(t_1, t_2) = R_{yy}(t_1, t_2) = 0 \).

For \( 0 < t_1 < t_2 \)

\[
R_{xy}(t_1, t_2) = 5\delta(t_1-t_2) e^{-2t_2} = 5e^{-2t_2}
\]

\[
R_{yy}(t_1, t_2) = \int_{0}^{t_1} 5e^{-2(t_1-\tau)} e^{-2(t_1-\tau)} d\tau = \frac{5}{4} e^{-2(t_2-t_1)} (1-e^{-4t_1})
\]

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9-33 \[ \int_{-\infty}^{\infty} e^{-\alpha \tau^2} e^{-st} d\tau = e^{s^2/4\alpha} \int_{-\infty}^{\infty} e^{-\alpha(\tau+s/2)} e^{-s^2/4\alpha} \\sqrt{\pi/\alpha} \] 

This yields

\[ e^{-\alpha \tau^2} \xrightarrow{\omega = \omega_0} \sqrt{\pi/\alpha} e^{-\omega_0^2/4\alpha} \]

\[ e^{-\alpha \tau^2} \cos \omega_0 \tau \xrightarrow{\omega = \omega_0} \frac{1}{2\sqrt{\alpha}} \left[ e^{-(\omega-\omega_0)^2/4\alpha} + e^{-(\omega+\omega_0)^2/4\alpha} \right] \]

9-34 

\[ G(x_1,x_2;\omega) = \int_{-\infty}^{\infty} f(x_1,x_2;\tau) e^{-j\omega \tau} d\tau \]

\[ R(\tau) = E[x(t+\tau)x(t)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1(x_2;\tau) f(x_1,x_2;\tau) dx_1 dx_2 \]

\[ S(\omega) = \int_{-\infty}^{\infty} R(\tau) e^{-j\omega \tau} d\tau = \int_{-\infty}^{\infty} e^{-j\omega \tau} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1(x_2;\tau) f(x_1,x_2;\tau) dx_1 dx_2 d\tau \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1(x_2) \int_{-\infty}^{\infty} e^{-j\omega \tau} f(x_1,x_2;\tau) d\tau dx_1 dx_2 \]

9-35 The process \( y(t) = x(t+a) - x(t-a) \) is the output of a system with input \( x(t) \) and system function \( H(\omega) = e^{j\omega a} - e^{-j\omega a} = 2j \sin \omega a \).

Hence [see (9-150)]

\[ S_y(\omega) = 4 \sin^2 \omega a S_x(\omega) = (2 - e^{j2\omega a} - e^{-j2\omega a}) S_x(\omega) \]

\[ R_y(\tau) = 2 R_x(\tau) - R_x(\tau+2a) - R_x(\tau-2a) \]
9-36 Since $S(\omega) \geq 0$, we conclude with (9-136) that

$$R(0) - R(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega)(1 - \cos \omega \tau) \, d\omega$$

$$\geq \frac{1}{8\pi} \int_{-\infty}^{\infty} S(\omega)(1 - \cos 2\omega \tau) \, d\omega = \frac{1}{4} [R(0) - R(2\tau)]$$

and the result follows for $n = 1$. Repeating the above, we obtain the general result.

9-37 From (6-197)

$$E(x^2(t + \tau)x^2(t)) = E(x^2(t + \tau))E(x^2(t)) + 2E^2(x^2(t + \tau)x^2(t))$$

Hence,

$$R_y(\tau) = R_x^2(0) + 2R_x^2(\tau) = \Gamma^2(1 + e^{-2\alpha|\tau|} + e^{-2\alpha|\tau|} \cos 2\beta \tau)$$

$$S_y(\omega) = \left[ 2\pi \delta(\omega) + \frac{4\alpha}{4a^2 + \omega^2} + \frac{2\alpha}{4a^2 + (\omega - 2\beta)^2} + \frac{2\alpha}{4a^2 + (\omega + 2\beta)^2} \right]$$

Furthermore,

$$\eta_y = E(x^2(t)) = R_x(0) \quad C_y(\tau) = 2R_x^2(\tau)$$

9-38

$$\int_{-\infty}^{\infty} S(\omega) \left\{ \sum_{i} a_i e^{j\omega t_i} \right\}^2 \, d\omega = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} S(\omega) \sum_{i,k} a_i a_k^* e^{j\omega(t_i - t_k)} \, d\omega$$

$$= \sum_{i,k} a_i a_k^* R(t_i - t_k) \geq 0$$

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(a) \[ S(s) = \frac{1}{1+s^4} = \frac{1}{(s^2 + \sqrt{2}s + 1)(s^2 - \sqrt{2}s + 1)} \]

A special case of example 9-27b with \( b = \sqrt{2}, c = 1 \). Hence,

\[ R(\tau) = \frac{1}{2\sqrt{2}} e^{-|\tau|/\sqrt{2}} \left( \cos \frac{\tau}{\sqrt{2}} + \sin \frac{|\tau|}{\sqrt{2}} \right) \]

(b) From the pair \( e^{-2|\tau|} \leftrightarrow 4/(4 + \omega^2) \) and the convolution theorem it follows that

\[ e^{-2|\tau|} * e^{-2|\tau|} \leftrightarrow \frac{16}{(4 + \omega^2)^2} \]

Hence, for \( \tau > 0 \)

\[ 16 R(\tau) = \int_{-\infty}^{\infty} e^{-2|x|} e^{-2|\tau-x|} \, dx = \int_{-\infty}^{0} e^{2x} e^{-2(\tau-x)} \, dx \]

\[ + \int_{0}^{\tau} e^{-2x} e^{-2(\tau-x)} \, dx + \int_{\tau}^{\infty} e^{-2x} e^{2(\tau-x)} \, dx = \frac{1}{2} e^{-2\tau} (1 + 2\tau) \]

And since \( R(-\tau) = R(\tau) \), the above yields

\[ e^{-2|\tau|} \frac{1 + 2|\tau|}{32} \leftrightarrow \frac{1}{(4 + \omega^2)^2} \]

\[ \begin{align*}
9-40 & \quad \begin{align*}
H^*(-s^*) \bigg|_{s=j\omega} &= H^*(j\omega) & H^*(1/z^*) \bigg|_{z=e^{j\omega T}} &= H^*(e^{j\omega T})
\end{align*}
\end{align*} \]

Hence

\[ \begin{align*}
H(s)H^*(-s^*) \bigg|_{s=j\omega} &= |H(j\omega)|^2 & H(z)H^*(1/z^*) \bigg|_{z=e^{j\omega T}} &= |H(e^{j\omega T})|^2
\end{align*} \]
9-41 From (6-197)

\[ R_y(\tau) = E\{x^2(t+\tau)\} \]
\[ = E\{x^2(t)\}E\{x^2(t)\} + 2 E^2\{x(t+\tau)x(t)\} = R_x^2(0) + 2 R_x^2(\tau) \]

From the above and the frequency convolution theorem it follows that

\[ S_y(\omega) = 2\pi R_x^2(0)\delta(\omega) + \frac{1}{\pi} S_x(\omega) * S_x(\omega) \]

\[ S_y(\omega) \]

\[ S_x(\omega) \]

\[ S_y(\omega) \]

---

9-42 \( \tilde{y}(t) = 2\tilde{x}(t) + 3\tilde{x}'(t) \quad \eta_x = 5 \quad C_{xx}(\tau) = 4e^{-2|\sigma|} \)

The process \( \tilde{y}(t) \) is the output of the system \( H(s) = 2+3s \) with input \( \tilde{x}(t) \). Hence,

\( \eta_y = 5H(0) = 10 \)

\[ S_{yy}(\omega) = S_{xx}(\omega) + 2(9) = \frac{16}{4+9\omega^2}(4+9\omega^2) = 144 - \frac{512}{4+9\omega^2} = S_{yy}(\omega) - 2\pi\rho^2\delta(\omega) \]
9-43 (a) \( y'(t) + 3y(t) = x(t) \), \( R_{xx}(r) = 5\delta(r) \). The process \( y(t) \) is the output of the system
\[
H(s) = \frac{1}{s+3}, \quad h(t) = e^{-3t}u(t)
\]
Hence, [see (9-100) and (9-150)]
\[
E(y^2(t)) = 5 \int_0^\infty e^{-6t}dt = \frac{5}{6}
\]
\[
S_{yy}(\omega) = \frac{5}{\omega^2+9}, \quad R_{yy}(r) = \frac{5}{6} e^{-3|t|}
\]
(b) As in Example 9-18:
\[
E(y^2(t)) = 5 \int_0^t e^{-6\alpha}d\alpha = \frac{5}{6} (1-e^{-6t}) \quad t > 0
\]
\[
R_{xy}(t_1,t_2) = 5e^{-2|t_2-t_1|}u(t_1)u(t_2)u(t_2-t_1)
\]

9-44 We shall show that: If \( x(t) \) is a complex process with autocorrelation \( R(r) \) and \(|R(r)|=R(0)\) for some \( r_1 \), then \( R(r)=e^{j\omega_0r}w(r) \) where \( w(r) \) is a periodic function with period \( r_1 \). Furthermore, the process \( \tilde{x}(t) = e^{-j\omega_0t}x(t) \) is MS periodic.

**Proof** Clearly, \( R(r_1) = R(0)e^{j\phi} \). With \( \omega_0 = \phi/r_1 \),
\[
R_{yy}(r) = E(\tilde{x}(t+r)\tilde{x}^*(t)) = R(r)e^{-j\omega r}
\]
Hence, \( R_{yy}(r_1)=e^{-j\omega_0r_1}R(r_1) = R(0) = R_{yy}(0) \). From this and (10-168) it follows that the function \( w(r) = R_{yy}(r) \) is periodic.
(a) The cross spectrum \( S_{XX}(\omega) = -j \text{sgn} \omega \) is an odd function. Hence,

\[
E\{ \tilde{x}(t) \tilde{x}'(t) \} = \frac{-1}{2\pi} \int_{-\infty}^{\infty} \text{sgn} \omega S_{XX}(\omega) d\omega = 0
\]

(b) The process \( \tilde{x}(t) \) is the output of the system

\((-j \text{sgn} \omega)(-j \text{sgn} \omega) = -1\)

with input \( x(t) \). Hence, \( \tilde{x}(t) = -x(t) \).

9-46 In general

\[
E\{ y^2(t) \} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_x(\omega)|H(\omega)|^2 d\omega
\]

\[
\leq |H(\omega_m)|^2 \frac{1}{2\pi} \int_{-\infty}^{\infty} S_x(\omega) d\omega = E\{x^2(t)\}|H(\omega_m)|^2
\]

where \( |H(\omega_m)| \) is the maximum of \( |H(\omega)| \). In our case,

\[
|H(\omega)|^2 = \frac{1}{(5-\omega)^2 + 4\omega^2}
\]

is maximum for \( \omega = \sqrt{3} \)

and \( |H(\omega_m)|^2 = 1/16 \). Hence \( E\{y^2(t)\} \leq 10/16 \) with equality if \( R_x(10) = 10 \cos \sqrt{3} \tau \) (Fig. b).
If $R_x(\tau) = e^{j\omega_0 \tau}$, then $S_x(\omega) = 2\pi \delta(\omega-\omega_0)$, hence, the integral of $S_x(\omega)$ equals zero in any interval not including the point $\omega = \omega_0$. From (9-182) it follows that the same is true for the integral of $S_{xy}(\omega)$. This shows that $S_{xy}(\omega)$ is a line at $\omega = \omega_0$ for any $\chi(t)$.

(a) As in (9-147) and (9-149)

\[ R_{yx}(\tau) = R_{xx}(\tau) \ast h(\tau) = \int_{-\infty}^{\infty} e^{j\alpha(\tau-\gamma)} h(\gamma) d\gamma = e^{j\alpha \tau} H(\alpha) \]

\[ R_{yy}(\tau) = R_{xx}(\tau) \ast \rho(\tau) = \int_{-\infty}^{\infty} e^{j\alpha(\tau-\gamma)} \rho(\gamma) d\gamma = e^{j\alpha \tau} |H(\alpha)|^2 \]

(b) As in (9-94) and (9-95)

\[ R_{yx}(t_1, t_2) = e^{-j\beta t_2} \int_{-\infty}^{\infty} e^{j\alpha(t_1-\gamma)} h(\gamma) d\gamma = e^{j\alpha t_2} H(\alpha) \]

\[ R_{yy}(t_1, t_2) = e^{-j\alpha t_1} H(\alpha) \int_{-\infty}^{\infty} e^{-j\beta(t_2-\gamma)} \rho(\gamma) d\gamma = e^{j\alpha t_1} H(\alpha) H^*(\beta) \]

because $h(t)$ is real and $H(-\beta) = H^*(\beta)$.

If $S_{xx}(\omega) S_{yy}(\omega) \equiv 0$ then $S_{xx}(\omega) = 0$ or $S_{yy}(\omega) = 0$ in any interval $(a, b)$. From this and (10-168) it follows that the integral of $S_{xy}(\omega)$ in any interval equals zero, hence, $S_{xy}(\omega) \equiv 0$. 

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This is the discrete-time version of theorem (9-162). From (9-163)

\[ E\left( (x[n+m+1] - x[n+m])x[n] \right) \leq E\left( |x[n+m+1] - x[n+m]|^2 \right) E\left( |x[n]|^2 \right) \]

\[ (R[m+1] - R[m])^2 \leq 2(R[0] - R[1])R[0] = 0 \]

Hence, \( R[m+1] = R[m] \) for any \( m \).

We shall show that

\[ 2 \frac{R^2[1]}{R[0]} - R[0] \leq R[2] \leq R[0] \quad (i) \]

The covariance matrix of the RVs \( x[n] \), \( x[n+1] \), and \( x[n+2] \) is non-negative [see (7-29)]:

\[
\begin{vmatrix}
\end{vmatrix} \geq 0
\]

This yields


The above is a quadratic in \( R[2] \) with roots

\[ R[0] \text{ and } -R[0] + 2 R^2[1]/R[0] \]

Since it is nonpositive, \( R[2] \) must be between the roots as in (i)

If \( x[n] = Ae^{j\omega T} \) then

\[ R_x[m] = A^2E\left( e^{j(m+n)\omega T} e^{-j\omega T} \right) = A^2 \int_{-\frac{\sigma}{2}}^{\frac{\sigma}{2}} e^{j\omega T} f(\omega) d\omega \]

But [see (9-194)]

\[ R[m] = \frac{1}{2\sigma} \int_{-\sigma}^{\sigma} S_x(\omega) e^{j\omega T} d\omega \]

hence, \( A^2f(\omega) = S_x(\omega) / 2\sigma \)
(a) If \( y(0) = y'(0) = 0 \), then \( y(t) \) is the output of a system with input \( x(t)U(t) \) and impulse response \( h(t) \) such that

\[
\begin{align*}
    h''(t) + 7h'(t) + 10h(t) &= \delta(t) \\
    h(0^-) &= h'(0^-) = 0
\end{align*}
\]

\[
    h(t) = \frac{1}{3} \left( e^{2t} - e^{-5t} \right) U(t)
\]

and with \( q(t) = 5 \ U(t) \), \( (9-100) \) yields

\[
    E(y^2(t)) = \frac{5}{9} \int_0^t (e^{2\tau} - e^{-5\tau})^2 d\tau
\]

(b) If \( y[-1] = y[-2] = 0 \), then \( y[n] \) is the output of a system with input \( x[n]U[n] \) and delta response \( h[n] \) such that

\[
    8h[n] - 6h[n-1] + h[n-2] = \delta[n] \\
    h[-1] = h[-2] = 0
\]

\[
    h[n] = \left( \frac{1}{2n+2} - \frac{1}{2n+3} \right) U[n]
\]

and with \( q[n] = 5 \ U[n] \), \( (10-176) \) yields

\[
    E(y^2[n]) = 5 \sum_{k=0}^{n} \left( \frac{1}{2k+2} - \frac{1}{2k+3} \right)^2
\]

---

9-54

\[
y[n] = x[n]h[n] \\
\hat{h}[n] = 2^{-n} U[n]
\]

(a) \( E(y^2[n]) = 5*2^{4n} U[n] = 0 \)

\[
    R_{xy}[m_1,m_2] = 5\delta[m_1-m_2] 2^{-m_2} U[m_2] = 5 2^{-m_2} U[m_2-m_1]
\]

\[
    R_{yy}[m_1,m_2] = 5 \times 2^{-m_2} U[m_2-m_1] 2^{-m_1} U[m_1]
\]

\[
    = \frac{20}{3} \times 2^{-|m_1-m_2|}
\]

The first equation follows from \( (9-190) \) with \( q[n] = 5 \); the second and third from \( (9-191) \) with \( R_{xx}[m_1,m_2] = 5 \delta[m_1-m_2] \).

(b) With \( R_{xx}[m_1,m_2] = 5 \delta[m_1-m_2] U[m_1] U[m_2] \), Prob. 9-25a yields the following: For \( m_1 \) or \( m_2 \), \( R_{xy}[m_1,m_2] = R_{yy}[m_1,m_2] = 0 \).

For \( 0 < m_1 < m_2 \)

\[
    R_{xy}[m_1,m_2] = 5 \delta[m_1-m_2] 2^{-m_2} = 5 \times 2^{-m_2}
\]

\[
    R_{yy}[m_1,m_2] = \sum_{k=0}^{m_1} 5 \times 2^{-k-2} = \frac{5}{3} 2^{-m_2} 2^{-m_1} (4 - 2^{-m_1})
\]
(a) \[ R_x[m_1, m_2] = q[m_1] \delta[m_1-m_2] \]

\[ E(s^2) = \sum_{n=0}^{N} \sum_{k=0}^{N} a_n a_k E[x[n]x[k]] \]

\[ = \sum_{n=0}^{N} \sum_{k=0}^{N} a_n a_k q[n] \delta[n-k] = \sum_{n=0}^{N} a_n^2 q[n] \]

(b) \[ R_x(t_1, t_2) = q(t_1) \delta(t_1-t_2) \]

\[ E(s^2) = \int_{0}^{T} \int_{0}^{T} a(t)a(\tau) E[x(t)x(\tau)] dt d\tau \]

\[ = \int_{0}^{T} \int_{0}^{T} a(t)a(\tau)q(t) \delta(t-\tau) d\tau dt = \int_{0}^{T} a^2(t)q(t) dt \]
CHAPTER 10

10-1

(a) If $\zeta(t)$ is a Poisson process as in Fig. 9-3a, then for a fixed $t$, $\zeta(t)$ is a Poisson RV with parameter $\lambda t$. Hence [see (5-119)] its characteristic function equals $\exp\{\lambda t(e^{i\omega} - 1)\}$.

(b) If $\zeta(t)$ is a Wiener process then $f(x,t)$ is $N(0,\sqrt{at})$. Hence [see (5-100)] its first order characteristic function equals $\exp\{-at\omega^2/2\}$.

10-2 For large $T$, $\zeta(t)$ and $\chi(t)$ can be approximated by two independent Wiener processes as in (10-52):

$$f_x(x,t) = \frac{1}{\sqrt{2\pi at}} \ e^{-x^2/2at} \quad f_y(y,t) = \frac{1}{\sqrt{2\pi at}} \ e^{-y^2/2at}$$

Hence, $\zeta(t)$ has a Rayleigh density [see (6-70)]. [Note. Exactly, $\zeta(t)$ is a discrete-type RV taking the values $\sqrt{m^2 + n^2}$ where $m$ and $n$ are integers]. The product $f_z(z,t)dz$ equals approximately the probability that $\zeta(t)$ is between $z$ and $z + dz$ provided that $dz \gg T$. 

---
The voltage $y(t)$ is the output of a system with input $n_e(t)$ and system function

$$H_1(s) = \frac{1}{LCS^2 + RCS + 1}$$

Hence,

$$S_v(\omega) = S_{n_e}(\omega) |H_1(j\omega)|^2 = \frac{2kTR}{(1 - \omega^2LC)^2 + R^2\omega^2}$$

Furthermore,

$$Z_{ab}(s) = \frac{R + Ls}{LCs^2 + RCS + 1} \quad \text{Re} Z_{ab}(j\omega) = \frac{R}{(1 - \omega^2LC)^2 + R^2\omega^2}$$

in agreement with (10-75).

The current $i(t)$ is the output of a system with input $y_e(t)$ and system function

$$H_2(s) = \frac{1}{R + Ls}$$

Hence,

$$S_i(\omega) = S_{n_e}(\omega) |H_2(j\omega)|^2 = \frac{2kTR}{R^2 + \omega^2L^2}$$

Furthermore (short circuit admittance)

$$Y_{ab}(s) = \frac{1}{R + LS} \quad \text{Re} Y_{ab}(j\omega) = \frac{2kTR}{R^2 + L^2\omega^2}$$

in agreement with (10-78).

The equation $m\ddot{x}(t) + f\dot{x}(t) = f_x(t)$ specifies a system with

$$H(s) = \frac{1}{ms^2 + fs} \quad h(t) = \frac{1}{f}(1 - e^{-ft/m})U(t)$$

and (9-100) yields

$$E(x^2(t)) = \frac{2kTf}{f^2} \int_0^t (1 - e^{-2\alpha t})^2 dt \quad \alpha = \frac{f}{2m}$$
As in Example 12-2, \(a\) and \(b\) are such that

\[
\chi(t) = a \chi(0) + b \nu(0) \quad y(0)
\]

This yields

\[
R_{xx}(\tau) = a R_{xx}(0) + b R_{x\nu}(0)
\]

\[
R_{x\nu}(\tau) = a R_{x\nu}(0) + b R_{\nu\nu}(0)
\]

where [see (10-163)]

\[
R_{xx}(\tau) = A e^{-\alpha \tau} (\cos \beta \tau + \frac{\alpha}{\beta} \sin \beta \tau)
\]

\[
R_{x\nu}(\tau) = R_{x\nu}(\tau) = A e^{-\alpha \tau} (\sin \beta \tau) \frac{\alpha + \beta \tau}{\beta}
\]

\[
R_{\nu\nu}(\tau) = R_{\nu\nu}(\tau) = A e^{-\alpha \tau} (\cos \beta \tau - \frac{\alpha}{\beta} \sin \beta \tau) \frac{\alpha + \beta \tau}{\beta}
\]

Inserting into (1) and solving, we obtain

\[
a = e^{-\alpha \tau} (\cos \beta \tau + \frac{\alpha}{\beta} \sin \beta \tau)
\]

\[
b = \frac{1}{\beta} e^{-\alpha \tau} \sin \beta \tau
\]

Finally,

\[
P = E\{[\chi(t) - a \chi(0) - b \nu(0)] \chi(t)\} = R_{xx}(0) - a R_{xx}(t) - b R_{x\nu}(t)
\]

\[
= \frac{2kT T \tau}{m^2} \left[1 - e^{-2\alpha \tau} (1 + \frac{2\alpha^2}{\beta} \sin^2 \beta \tau + \frac{\alpha}{\beta} \sin 2\beta \tau)\right]
\]

10-6 If \(\bar{\chi}(t) = \bar{w}(t^2)\) then [see (10-70)]

\[
R_{\chi}(t_1, t_2) = E\{\bar{w}(t_1^2) \bar{w}(t_2^2)\} = \alpha_1 t^2
\]

If \(\bar{y}(t) = \bar{w}(t)\) then [see (6-197)]

\[
R_{\gamma}(t_1, t_2) = E\{\bar{w}(t_1)^2 \bar{w}(t_2)^2\}
\]

\[
= E \bar{w}(t_1) E\{\bar{w}(t_2)^2\} + 2 E \bar{w}(t_1) E\{\bar{w}(t_2)\} = \alpha_2 t_1 t_2 + 2 \alpha_2 t_1^2
\]
From (10-112):

\[ \eta_s = 3 \int_0^{10} 2 \, dt = 60 \quad \sigma_s^2 = 3 \int_0^{10} 4 \, dt = 120 \quad E(s^2) = 3720 \]

\( s(7) = 0 \) if there are no points in the interval (7-10, 7). The number of points in this interval is a Poisson RV with parameter \( 10 \lambda = 30 \). Hence, \( P(s(7) = 0) = e^{-30} \).

\[ H(\omega) = jB(\omega) \]

From the assumption: \( S_{xx}(\omega) = S_{yy}(\omega) \quad S_{xy}(-\omega) = -S_{xy}(\omega) \)

From (9-148): \( S_{yy}(\omega) = S_{xx}(\omega)|H(\omega)|^2 \quad S_{xy}(\omega) = S_{xx}(\omega)H^*(\omega) \)

Combining, we obtain

\[ |H(\omega)|^2 = 1 \quad H(-\omega) = -H(\omega) \]

Since \( h(t) \) is real, the second equation yields \( H(\omega) = jB(\omega) \) and from the first it follows that

\[ |B(\omega)| = 1 \]

as in the figure.
10-9 With \( i(t) = a(t) \), \( q(t) = b(t) \), (11-63) yields

\[
S_i(\omega) = S_q(\omega) \quad S_{iq}(\omega) = -S_{qi}(\omega) = S_{qi}(-\omega)
\]

Hence [see (11-75) and (11-82)],

\[
S_w(\omega) = 2S_i(\omega) + 2jS_{qi}(\omega)
\]
\[
S_w(-\omega) = 2S_i(\omega) - 2jS_{qi}(\omega)
\]

Adding and subtracting, we obtain

\[
4S_i(\omega) = S_w(\omega) + S_w(-\omega) \quad 4jS_{iq}(\omega) = S_w(-\omega) - S_w(\omega)
\]

10-10 From (10-133)

\[
x(t) = \text{Re} \left[ w(t) e^{j\omega_0 t} \right]
\]

\[
x(t-\tau) = \text{Re} \left[ w(t-\tau) e^{j\omega_0 t} \right] = \text{Re} \left[ w(t-\tau) e^{j\omega_0 (t-\tau)} \right]
\]

\[
w_{\tau}(t) = w(t-\tau) e^{-j\omega_0 \tau}
\]

10-11 \( R''_x(\tau) \leftrightarrow -\omega^2 S_x(\omega) \)

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} \omega^2 S_x(\omega) d\omega = -R''_x(0)
\]

and with \( \omega_0 \) the optimum carrier frequency, (10-150) yields

\[
E\{ |w'(t)|^2 \} = \frac{M}{2\pi} = -2R''_x(0) - 2\omega_0^2 R_x(0)
\]
From the stationarity of the process $x(t) \cos \omega t + y(t) \sin \omega t$ it follows that [see (10-130)]

$$C_{xx}(r) = C_{yy}(r) \quad C_{xy} = - C_{yx}(r)$$ (i)

Using these identities, we shall express the joint density $f(X,Y)$ of the 2n RVs

$$X = [x(t_1), \ldots, x(t_n)] \quad Y = [y(t_1), \ldots, y(t_n)]$$

in terms of the covariance matrix $C_{ZZ}$ of the complex vector $Z = \tilde{X} + j \tilde{Y}$. From (i) it follows that

$$E(x(t_i) x(t_j)) = E(y(t_i) y(t_j)) \quad E(x(t_i) y(t_j)) = - E(y(t_i) x(t_j))$$

This yields

$$C_{XX} = C_{YY}, \text{ and } C_{XY} = - C_{YX}; \text{ hence, } f(X,Y) \text{ is given by (8-62).}$$

The signal $\zeta(t) = f(t)$ is an extreme case of a cyclostationary process as in (10-178) with

$$h(t) = \begin{cases} f(t) & 0 \leq t \leq T \\ 0 & \text{otherwise} \end{cases} \quad H(\omega) = \int_0^T f(t) e^{-j\omega t} dt$$

and $\zeta_m = 1, R[m] = 1$. Hence [see (10A-2)]

$$\sum_{m=-\infty}^{\infty} R_m e^{-j\omega T} = \sum_{m=-\infty}^{\infty} e^{-j\omega T} = T \sum_{m=-\infty}^{\infty} \delta(\omega - \frac{2\pi}{T} m)$$

From the above and (10-180) it follows that the process $x(t) = f(t - \theta)$ is stationary with power spectrum

$$S(\omega) = \left| \int_0^T f(t) e^{-j\omega t} dt \right|^2 \sum_{m=-\infty}^{\infty} \delta(\omega - \frac{2\pi}{T} m)$$
The process

\[ y_N(t) = x(t + \tau) - \sum_{n=-N}^{N} x(t + nT) \frac{\sin(\pi - nT)}{\sigma(\pi - nT)} \]

is the output of a system with input \( x(t) \) and system function

\[ H_N(\omega) = e^{j\omega\tau} - \sum_{n=-N}^{N} \frac{\sin(\pi - nT)}{\sigma(\pi - nT)} e^{jn\omega} \]

Furthermore, \( \varepsilon_N(\tau) = y_N(0) \), hence [see (9-153)]

\[ E\{\varepsilon_N^2(\tau)\} = E\{y_N^2(0)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) |H_N(\omega)|^2 d\omega \]

The function \( H_N(\omega) \) is the truncation error in the Fourier series expansion of \( e^{j\omega \tau} \) in the interval \((-\sigma, \sigma)\). Hence, for \( N > N_0 \)

\[ |H_N(\omega)| < \varepsilon \quad |\omega| < \sigma \]

From this and (1) it follows that, if \( S(\omega) = 0 \) for \( |\omega| < \sigma \), then

\[ E\{\varepsilon_N^2(\tau)\} = \frac{1}{2\pi} \int_{-\sigma}^{\sigma} S(\omega) |H_N(\omega)|^2 d\omega < \varepsilon R(0) \quad N > N_0 \]
10-15 [see after (10-195)]

\[ R(0) - R(\tau) = \frac{1}{2\pi} \int_{-\sigma}^{\sigma} S(\omega)(1 - \cos \omega \tau) d\omega \]

\[ \leq \frac{2}{4\pi} \int_{-\sigma}^{\sigma} \omega^2 S(\omega) d\omega = \frac{-\tau^2}{2} R''(0) \]

Furthermore, since

\[ \sin \phi \geq \frac{2\phi}{\pi} \quad 0 \leq \phi \leq \frac{\pi}{2} \]

we obtain

\[ R(0) - R(\tau) = \frac{1}{2\pi} \int_{-\sigma}^{\sigma} S(\omega) 2 \sin \frac{\omega \tau}{2} d\omega \]

\[ \geq \frac{2\tau^2}{\pi} \frac{1}{2\pi} \int_{-\sigma}^{\sigma} \omega^2 S(\omega) d\omega = \frac{-2\tau^2}{2} \frac{\pi}{\pi} R''(0) \]

10-16 With \( T = \pi/\sigma \)

\[ R(mT) = E[x(nT + mT)x(nT)] = \begin{cases} I & \text{m = 0} \\ \eta^2 & \text{m \neq 0} \end{cases} \]

Hence [see (10-196)]

\[ R(\tau) = \sum_{m=-\infty}^{\infty} R(mT) \frac{\sin \sigma(\tau - mT)}{\sigma(\tau - mT)} = \eta^2 + (I - \eta^2) \frac{\sin \tau}{\pi T} \]

\[ S(\omega) = 2\pi \eta^2 \delta(\omega) + 2\pi (I - \eta^2) p_\sigma(\omega) \]
10-17 Given \( E(x(n+m)x(n)) = N\delta[m] \)

This is a special case of Prob. 10-16 with \( \eta = 0, I = N \).

10-18 If \(|\tau| < \pi/(2\sigma)\), then

\[
\cos \omega \tau \geq \cos \sigma \tau \quad |\omega| \leq \sigma
\]

\[
R(\tau) = \frac{1}{2\pi} \int_{-\sigma}^{\sigma} S(\omega) \cos \omega \tau d\omega
\]

\[
\geq \frac{\cos \sigma \tau}{2\pi} \int_{-\sigma}^{\sigma} S(\omega) d\omega = R(0) \cos \sigma \tau
\]

10-19 From (10-133) with \( c = \sigma \)

\[
P_1(\omega, \tau) + j\omega P_2(\omega, \tau) = 1
\]

\[
P_1(\omega, \tau) + j(\omega+\tau)P_2(\omega, \tau) = e^{j\sigma \tau}
\]

Hence,

\[
P_1(\omega, \tau) = 1 - \frac{\omega}{\sigma} (e^{j\sigma \tau} - 1) \quad P_2(\omega, \tau) = \frac{1}{j\sigma} (e^{j\sigma \tau} - 1)
\]

Inserting into (11-141), we obtain

\[
p_1(\tau) = \frac{4 \sin^2 (\sigma \tau/2)}{\tau^2} \quad p_2(\tau) = \frac{4 \sin^2 (\sigma \tau/2)}{\sigma^2 \tau}
\]

and with \( t = 0 \), the desired result follows from (10-206) because \( \tau = 2T \) and

\[
\sin^2 \frac{\sigma(\tau-2nT)}{2} = \sin^2 \left( \frac{\sigma T}{2} - n\pi \right) = \sin^2 \frac{\sigma T}{2}
\]
10-20 As in (10-213)
\[
\begin{align*}
\mathcal{P}(\omega) &= \frac{1}{\lambda} \int_{-a}^{a} \cos \omega t \, z(t) \cos \omega_c t \, dt \\
E\{\mathcal{P}(\omega)\} &= \int_{-a}^{a} \cos \omega t \, \cos \omega_c t \, dt \\
\sigma_P^2(\omega) &= \frac{1}{\lambda} \int_{-a}^{a} \cos^2 \omega_c t \, 2 \cos^2 \omega t t_2 \, dt_2
\end{align*}
\]

10-21 We shall show that if
\[
\begin{align*}
\tilde{x}(\omega) &= \frac{1}{\lambda} \sum_{|t_j| < c} x(t_j) e^{-j\omega t_j} = \frac{1}{\lambda} \int_{-a}^{a} \tilde{x}(t) e^{-j\omega t} \, dt
\end{align*}
\]

where \(\tilde{x}(t) = \sum \delta(t-t_j)\) is a Poisson impulse train, then
\[
E(|\tilde{x}(\omega)|^2) \approx 2cS_x(\omega) + \frac{2c}{\lambda} R_x(0)
\]

Proof

Since \(R_x(r) = \lambda^2 + \lambda\delta(r)\), it follows that
\[
\begin{align*}
E \left\{ |\tilde{x}(\omega)|^2 \right\} &= \frac{1}{\lambda^2} \int_{-c}^{c} \int_{-c}^{c} R_x(t_1-t_2) e^{-j\omega(t_1-t_2)} dt_1 dt_2 \\
&= \int_{-c}^{c} e^{j\omega t_2} \int_{-c}^{c} R_x(t_1-t_2) e^{-j\omega t_1} dt_1 dt_2 + \frac{1}{\lambda} \int_{-c}^{c} R_x(0) dt_2
\end{align*}
\]

If \(\int_{-\infty}^{\infty} |R_x(r)| < \infty\) then for sufficient large \(c\), the inner integral on the right is nearly equal to \(S_x(\omega)^{-j\omega t_2}\) and (i) follows.
\[ E(\tilde{z}(t)) = g(t) \quad E(\tilde{w}(t)) = g(t) - g(T)T/T = g(t) \]

\[ w(t) = (1 - \frac{t}{T}) \int_0^t x(\alpha) d\alpha - \frac{t}{T} \int_0^T x(\alpha) d\alpha \]

The above two integrals are uncorrelated because \( \tilde{y}(t) \) is white noise. Hence, as in Example 9-5

\[ \sigma_w^2 = (1 - \frac{t}{T})^2 \frac{Nt}{T} + \frac{t^2}{T^2} N(T - t) = Nt(1 - \frac{t}{T}) \]

Note: The above shows that the information that \( g(T) = 0 \) can be used to improve the estimate of \( g(t) \). Indeed, if we use \( \tilde{w}(t) \) instead of \( \tilde{z}(t) \) for the estimate of \( g(t) \) in terms of the data \( \tilde{x}(t) \), the variance is reduced from \( Nt \) to \( Nt(1-t/T) \).

10-23

(a) Since \[ \sum_{i} |a_i b_i| \leq \sum |a_i| |b_i| \], it suffices to assume that the numbers \( a_i \) and \( b_i \) are real. The quadratic

\[ I(z) = \sum_i (a_i - z b_i)^2 = z^2 \sum_i b_i^2 - 2z \sum_i a_i b_i + \sum_i a_i^2 \]

is nonnegative for every real \( z \), hence, its discriminant cannot be positive. This yields (i).

(b) With \( f[n] \) and \( R_v[m] = S_0 \delta[m] \) as in Prob. 10-24a (white noise)

\[ y_f[n_0] = \sum h[n] f[n_0-n] \quad y_v[n] = \sum h[n] y[n] \]

\[ E(y_v^2[n]) = S_0 \delta[0] = S_0 \sum |h[n]|^2 \]

[see (9-213)] And (i) yields

\[ \frac{y_f^2[n_0]}{E(y_v^2[n])} = \frac{\sum |h[n] f[n_0-n]|^2}{S_0 \sum h^2[n]} \leq \frac{1}{S_0} \sum |h[n]|^2 \]

with equality iff \( h[n] = kf_{*}[n_0-n] \).
10-24 (a) Given $F(z)$ and $S_y(\omega) = S_0 \equiv$ constant. The $z$ transform of $y_f[n]$ equals $F(z)H(z)$. Hence, [see (9-109)]

$$y_f[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(e^{j\omega T})H(e^{j\omega T})e^{jn\omega T}d\omega$$

$$y_f^2[n] = \left| \int_{-\pi}^{\pi} F(e^{j\omega T})H(e^{j\omega T})d\omega \right|^2$$

$$\frac{y_f^2[n]}{E\{y_v^2[n]\}} = \frac{S_0 \int_{-\pi}^{\pi} \left| H(e^{j\omega T}) \right|^2 d\omega}{S_0}$$

$$\leq \frac{1}{S_0} \int_{-\pi}^{\pi} \left| F(e^{j\omega T}) \right|^2 d\omega$$

The last inequality follows from Schwarz's inequality with equality iff

$$H(e^{j\omega T}) = kF^*(e^{j\omega T}) = kF(e^{-j\omega T}), \text{ i.e., iff } H(z) = kF(z^{-1})$$

(b) Given arbitrary $R_m$, $F(z)$, and the form of $H(z)$ (FIR); to find the coefficients $a_m$ of $H(\omega)$. In this case

$$y_f[n] = a_0f[n] + a_1f[n-1] + \cdots + a_Nf[n-N]$$

$$y_v[n] = a_0v[n] + a_1v[n-1] + \cdots + a_Nv[n-N]$$

To maximize the signal-to-noise ratio it suffices to minimize

$$E\{y_v^2[n]\} = \sum_{k,r=0}^{N} a_k a_r R_{\nu}[k-r]$$

subject to the constraint that the sum

$$y_f[0] = a_0f[0] + a_1f[-1] + \cdots + a_Nf[-N]$$

is constant. With $\lambda$ a constant (Lagrange multiplier), we minimize the sum
\[ I = \sum_{k, r=0}^{N} a_k a_R[k-r] - \lambda \left[ \sum_{k=0}^{N} a_k f[-k] - y_f[0] \right] \]

does not yield the system

\[ \frac{\partial I}{\partial a_k} = 0 = \sum_{r=0}^{N} \left[ a_R[k-r] - \lambda f[-k] \right] \quad k = 0, \ldots, N \]

whose solution yields \( a_k \).

\[ B = A|H(\omega_0)| = \frac{A}{\sqrt{a^2 + \omega_0^2}} \quad S_{y_n}(\omega) = \frac{N}{a^2 + \omega^2} \]

\[ R_{y_n}(\tau) = \frac{N}{2a} e^{-a|\tau|} \quad \mathbb{E}[y_n^2(t)] = R_{y_n}(0) = \frac{N}{2a} \]

\[ \frac{B^2}{\mathbb{E}[y_n^2(t)]} = \frac{2A^2}{N} \frac{\alpha}{a^2 + \omega_0^2} \quad \text{Max. if } \alpha = \omega_0 \]
Since \( H(\omega) \) is determined within a constant factor, we can assume that the response \( y_\tau(t_o) \) of the optimum \( H(\omega) \) due to \( f(t) \) is constant:

\[
y_\tau(t_o) = \sum_{i=0}^{m} a_i f(t_o-iT) = c
\]  

(i)

Our problem is to minimize the variance

\[
V = E(y_\phi^2(t)) = \sum_{n=0}^{m} a_n \sum_{i=0}^{m} a_i R(nT-iT)
\]  

(ii)

of \( y_\phi(t) \) subject to the constraint (i). This yields the system

\[
\frac{\partial V}{\partial a_n} = \sum_{i=0}^{m} a_i R(nT-iT) - k f(t_o-nT) = 0
\]

where \( k \) is a constant (lagrange multiplier). With \( a_n \) so determined, we conclude from (ii) that

\[
V = \sum_{n=0}^{m} ka_n f(t_o-nT) = ky_\tau(t_o) \\
r^2 = \frac{y_\tau^2(t_o)}{ky_\tau(t_o)}
\]

--------------------------

10-27 \( R_{yy}(\mu, \nu) = E(\xi(t+\mu)+c[\xi(t+\nu)+c[\xi(t)+c]) = R(\mu, \nu) + cR(\mu) + cR(\nu) + cR(\mu-\nu) + c^3 \)

because \( E(\xi(t)) = 0 \). Furthermore,

\[
R(\mu) \leftrightarrow 2\pi S(u)\delta(v) \\
R(\nu) = 2\pi \delta(u)S(v) \\
c^3 \leftrightarrow 4\pi^2 \delta(u)\delta(v)
\]

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R(\mu-\nu)e^{-j(u\mu+v\nu)}d\mu d\nu = \int_{-\infty}^{\infty} R(r)e^{-jur}dr \int_{-\infty}^{\infty} e^{-j(u+v)\nu}d\nu = 2\pi S(u)\delta(u+v)
\]

-----------------------------------
We shall use the equations $E(\tilde{x}(t)) = 0$, $E(\tilde{x}^2(t)) = \lambda t$. Suppose that $t_1 < t_2 < t_3$.

Clearly,

$$\tilde{x}(t_2) = \tilde{x}(t_1) + [\tilde{x}(t_2) - \tilde{x}(t_1)]$$

$$\tilde{x}(t_3) = \tilde{x}(t_1) + [\tilde{x}(t_2) - \tilde{x}(t_1)] + [\tilde{x}(t_3) - \tilde{x}(t_2)]$$

Inserting into the product $\tilde{x}(t_1)\tilde{x}(t_2)\tilde{x}(t_3)$ and using the identity $E(\tilde{x}(t_1) - \tilde{x}(t_2)) = 0$ and the independence of the three terms on the right of (i), we obtain

$$E(\tilde{x}(t_1)\tilde{x}(t_2)\tilde{x}(t_3)) = E(\tilde{x}^3(t_1)) = \lambda t_1 = \lambda \min(t_1, t_2, t_3)$$

Since $\tilde{x}(t) = \tilde{x}'(t)$, we conclude from (9-120)–(9-122) that

$$R_{\tilde{x}\tilde{x}}(t_1, t_2, t_3) = \frac{\partial^3 R_{xx}(t_1, t_2, t_3)}{\partial t_1 \partial t_2 \partial t_3} = \lambda \frac{\partial^3 \min(t_1, t_2, t_3)}{\partial t_1 \partial t_2 \partial t_3}$$

It suffices therefore to show that the right side equals $\lambda \delta(t_1-t_2)\delta(t_1-t_3)$. This is a consequence of the following:

$$\frac{\partial \min(t_1, t_2, t_3)}{\partial t_3} = t_1 U(t_2-t_1)\delta(t_3-t_1) + t_2 U(t_1-t_2) \delta(t_3-t_2)$$

$$+ U(t_1-t_3)U(t_2-t_3)\delta(t_1-t_3)U(t_2-t_3)\delta(t_1-t_3)U(t_1-t_3)\delta(t_2-t_3)$$

$$= U(t_1-t_3)U(t_2-t_3)$$

because $t_3 \delta(t_1-t_3) = t_3 \delta(t_3-t_3)$. Hence,

$$\frac{\partial^2 \min(t_1, t_2, t_3)}{\partial t_2 \partial t_3} = U(t_1-t_3)\delta(t_2-t_3) \quad \frac{\partial^2 \min(t_1, t_2, t_3)}{\partial t_1 \partial t_2 \partial t_3} = \delta(t_1-t_2)\delta(t_1-t_3)$$

See outline given in text.
CHAPTER 11

11-1

\[ S_x(z) = \frac{5 - 2(z + 1/z)}{10 - 3(z + 1/z)} = \frac{2}{3} + \frac{5/9}{10/3 - (z + 1/z)} \]

\[ R[m] = \frac{2}{3} + \frac{5}{18} 3^{-|m|} \quad \Gamma(z) = \frac{3z - 1}{2z - 1} \]

11-2

\[ S_x(s) = \frac{s^4 + 64}{s^4 - 10s^2 + 9} = \frac{s^2 + 4s + 8}{s^2 + 4s + 3} \frac{s^2 - 4s + 8}{s^2 - 4s + 3} \]

\[ L(s) = \frac{s^2 + 4s + 8}{s^2 + 4s + 3} \]

11-3

**First proof**

\[ g[n] = \sum_{k=0}^{\infty} \ell[n] \delta[n-k] \quad \text{E}\{x^2[n]\} = \sum_{k=0}^{\infty} \ell^2[k] \]

**Second proof**

\[ S(z) = L(z) L(1/z) \quad R[m] = \ell[m] * \ell[-m] = \sum_{k=0}^{\infty} \ell[k] \ell[k-m] \]

\[ R[0] = \sum_{k=0}^{\infty} \ell^2[k] \]
11-4 (a) This is a special case of (11-22) and (11-23).

(b) From (a) it follows that

\[ R_{yx}''(\tau) + 3 R_{yx}'(\tau) + 2 R_{yx}(\tau) = q(\delta(\tau)) \]

Since \( R_{xx}(\tau) = 0 \) for \( \tau < 0 \), the above shows that

\[ R_{yx}(\tau) = 0 \] for \( \tau < 0^- \) \hspace{1cm} R_{yx}'(0^-) = 0

Furthermore,

\[ S_{yx}(s) = \frac{q}{s^2 + 3s + 2} \]

hence (initial value theorem)

\[ R_{yx}(0^+) = \lim_{s \to \infty} s \cdot S_{yx}(s) = 0 \hspace{1cm} R_{yx}'(0^+) = \lim_{s \to \infty} s^2 S_{yx}(s) = q \]

Similarly,

\[ R_{yy}''(\tau) + 3 R_{yy}'(\tau) + 2 R_{yy}(\tau) = R_{xy}(\tau) = R_{yx}(-\tau) = 0 \] for \( \tau > 0 \)

\[ S_{yy}(s) = \frac{q}{(s^2 + 3s + 2)(s^2 - 3s + 2)} = \frac{qs/12 + q/4}{s^2 + 3s + 2} + \frac{-q}{s^2 - 3s + 2} \]

\[ S_{yy}^+(s) = \frac{qs/12 + q/4}{s^2 + 3s + 2} \]

\[ R_{yy}(0^+) = R_{yy}(0) = \lim_{s \to \infty} s^2 S_{yy}^+(s) = \frac{q}{12} \]

\[ R_{yy}'(0) = \lim_{s \to \infty} s \left[ S_{yy}^+(s) - \frac{q}{12} \right] = 0 \]

11-5

\[ S_x(z) = \frac{S_x(z) + S_y(z) = \frac{1}{D(z)} + q = \frac{1+qD(z)}{D(z)}} \]

If \( R_s[m] = 2^{-|m|} \) and \( S_{x^+}(z) = 5 \), then (see Example 9-31)

\[ S_s(z) = \frac{1.5}{2.5 - (z^{-1} + z)} \hspace{1cm} S_x(z) = \frac{5-14z^{-1} + 5z^{-2}}{1-2.5z^{-1} + z^{-2}} \]
The process

\[ y[n] = \frac{1}{n} \sum_{k=1}^{n} x(nT + kT) \]

is the output of a system with input \( x[n] \) and system function

\[ H(z) = \frac{1}{n} \sum_{k=1}^{n} z^k \]

Furthermore, \( s = x(0) \) and

\[ n^2 |H(e^{j\omega T})|^2 = \left| \sum_{k=1}^{n} e^{j\omega T} \right|^2 \]

\[ = \left| \frac{e^{j\omega T} - e^{j(n+1)\omega T}}{1 - e^{j\omega T}} \right|^2 = \frac{\sin^2 n\omega T/2}{\sin^2 \omega T/2} \]

Hence [see (9-51)]

\[ E\{s^2\} = R_y[0] = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_x(\omega) \frac{\sin^2 n\omega T/2}{\sin^2 \omega T/2} d\omega \]

11-7

Since \( R(t_1, t_2) = e^{-c|t_1-t_2|} \), (12-58) yields

\[ \int_{-a}^{t_1} e^{-c(t_1-t_2)} \phi(t_2) dt_2 + \int_{t_1}^{a} e^{-c(t_1-t_2)} \phi(t_2) dt_2 = \lambda \phi(t_1) \quad (i) \]

Differentiating twice and using (i) we obtain (omitting details)

\[ \lambda \phi''(t) + (2c - \lambda c^2) \phi(t) = 0 \]

Hence;

\[ \phi(t) = \beta \cos \omega t \quad \text{and} \quad \phi(t) = \beta' \cos \omega't \]

To determine \( \omega \), we insert into (i). This yields

\[ \frac{2c}{c^2 + \omega^2} + \frac{\omega \sin \omega \lambda}{c^2 + \omega^2} \left( e^{c t} + e^{-c t} \right) = 2c \lambda \cos \omega t \]
This yields

\[ \omega_n \sin \alpha \omega_n - c \cos \alpha \omega_n = 0 \quad \lambda_n = \frac{2c}{c^2 + \omega_n^2} \]

The constants \( \beta_n \) are determined from (normalization)

\[ 1 = \int_{-a}^{a} \beta_n^2 \cos^2 \omega_n t \, dt \quad \beta_n^2 = \frac{1}{a + c \lambda_n} \]

Similarly for \( \beta_n' \sin \omega_p t \).

\[ \text{11-8 As in (9-60)} \]

\[ E(\left| X_T(\omega) \right|^2) = \int_{-T/2}^{T/2} R(t_1 - t_2) \frac{-j\omega(t_1 - t_2)}{T} \, dt_1 \, dt_2 \]

\[ = \int_{-T}^{T} (T - |\tau|) R(\tau) \, e^{-j\omega \tau} \, d\tau \]

Differentiating with respect to \( T \) and using the fact that if

\[ \phi(t) = \int_{-T}^{T} f(x; t) \, dx \]

then

\[ \frac{d\phi(t)}{dt} = f(t; t) - f(-t, t) + \int_{-T}^{T} \frac{\partial f}{\partial t}(x, t) \, dx \]

we obtain

\[ \frac{\partial E(\left| X_T(\omega) \right|^2)}{\partial T} = \int_{-T}^{T} R(\tau) \frac{-j\omega \tau}{T} \, d\tau = E\left( \frac{\partial}{\partial T} \left| X_T(\omega) \right|^2 \right) \]

The above approaches \( S(\omega) \) as \( T \to \infty \).
11-9
\[ E(\tilde{x}(\omega)) = \int_{-a}^{a} 5 \cos 3 t e^{-j\omega t} dt = \frac{5 \sin a(\omega-3)}{\omega-3} + \frac{5 \sin a(\omega+3)}{\omega+3} \]

Var. \( \tilde{x}(\omega) = 2a \cdot 4a. \)

11-10
\[ E(\tilde{x}(u)\tilde{x}(v)) = \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \sigma_n^2 \delta[n-k] e^{-j(nu-kv)T} \]
\[ = \sum_{n=-\infty}^{\infty} \sigma_n^2 e^{-jn(u-v)T} \]

11-11 Shifting the origin, we set
\[ \tilde{c}_n = \frac{1}{T} \int_{-T/2}^{T/2} \tilde{x}(t)e^{-jnw_0 t} dt \]
\[ \beta_n(\alpha) = \frac{1}{T} \int_{-T/2}^{T/2} R(r-\alpha)e^{-jn\omega_0 r} dr \]

(a) We shall show that if
\[ \tilde{x}(t) = \sum_{n=-\infty}^{\infty} \tilde{c}_n e^{jn\omega_0 t} \text{ then } E(\tilde{x}(t)\tilde{x}(t)^*) = 0 \text{ for } |t| < T/2 \] (i)

Proof
\[ E(\tilde{c}_n\tilde{x}_n^*(\alpha)) = \frac{1}{T} \int_{-T/2}^{T/2} E(\tilde{x}(t)\tilde{x}_n^*(\alpha))e^{-jn\omega_0 t} dt = \beta_n(\alpha) \]

The functions \( \beta_n(\alpha) \) are the coefficients of the Fourier expansion of \( R(r-\alpha) \):
\[ R(r-\alpha) = \sum_{n=-\infty}^{\infty} \beta_n(\alpha)e^{jnw_0 r} \quad |r| < T/2 \] (ii)

Hence
\[ E(\tilde{x}(t)\tilde{x}_n^*(t)) = \sum_{n=-\infty}^{\infty} E(\tilde{c}_n\tilde{x}_n^*(t))e^{jnw_0 t} = \sum_{n=-\infty}^{\infty} \beta_n(t)e^{jnw_0 t} \]
From (ii) it follows with \( r = \alpha = t \) that the last sum equals \( R(0) \). Similarly, \( E(\hat{x}(t)\hat{x}(t)) = R(0) \) and (i) results.

(b) \( E(\sum_{n} c_{m}^{*}) = \frac{1}{T} \int_{-T/2}^{T/2} E(\sum_{n} c_{m}^{*}(t)) e^{j\omega_{0}t} dt = \frac{1}{T} \int_{-T/2}^{T/2} \beta(t) e^{j\omega_{0}t} dt \)

(c) If \( T \) is sufficiently large, then

\[
T \beta(t) = \int_{-T/2}^{T/2} R(\tau - t) e^{-j\omega_{0}\tau} d\tau \approx S(n\omega_{0}) e^{-j\omega_{0}t} \\
E(\sum_{n} c_{m}^{*}) = \frac{1}{T} \int_{-T/2}^{T/2} e^{j(m-n)\omega_{0}t} d\alpha = \begin{cases} S(n\omega_{0})/T & m-n \vspace{1em} \text{for large } T \vspace{1em} \\
0 & m \neq n \end{cases}
\]

Thus, for large \( T \), the coefficients \( c_{n} \) of an arbitrary WSS process are nearly orthogonal.

11-12

\[
E(\hat{x}(t_{1})\hat{x}(t_{2})) = \frac{1}{4\pi^{2}} E \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E(\hat{x}(u)\hat{x}(v)) e^{ju(t_{1} - t_{2})} dudv \right\} \\
= \frac{1}{4\pi^{2}} E \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} Q(u)\delta(u - v) e^{ju(t_{1} - t_{2})} dudv \right\} = \frac{1}{4\pi^{2}} \int_{-\infty}^{\infty} Q(u) e^{ju(t_{1} - t_{2})} du
\]

This depends only on \( r = t_{1} - t_{2} \):

\[
R_{xx}(r) = \frac{1}{x^{2}} \int_{-\infty}^{\infty} Q(u) e^{jru} du \quad S_{xx}(\omega) = \frac{Q(\omega)}{2\pi}
\]

11-13 Equations (11-79) can be written in the following form:

\[
E(\hat{A}(\omega)\hat{A}(\omega)) = Q(\omega) \delta(u - \omega) = E(\hat{B}(\omega)\hat{B}(\omega)) \quad E(\hat{A}(\omega)\hat{B}(\omega)) = 0
\]

for \( u \geq 0, \ v \geq 0 \). We shall show that if the above is true and \( E(\hat{A}(\omega)) = E(\hat{B}(\omega)) = 0 \), then the process

\[
x(t) = \int_{0}^{\infty} \left[ A(\omega) \cos \omega t - B(\omega) \sin \omega t \right] d\omega
\]

is WSS.

**Proof** Clearly, \( E(x(t)) = 0 \) and
\[ E(\sim(t+r)\sim(t)) \]
\[ = \frac{1}{\pi^2} \int_0^\infty \int_0^\infty E(\sim(u)\cos(u) + B(u)\sin(u)) [A(v)\cos(v) - B(v)\sin(v)] \, du \, dv \]
\[ = \frac{1}{\pi} \int_0^\infty \int_0^\infty Q(u)\delta(u-v) [\cos(u)\cos(v) + \sin(u)\sin(v)] \, dv \, du \]
\[ = \frac{1}{\pi^2} \int_0^\infty Q(u) [\cos(u)\cos(v) + \sin(u)\sin(v)] \, du \]
\[ = \frac{1}{\pi^2} \int_0^\infty Q(u) \cos(u) du \]

From this and (9-136) it follows that \( \sim(t) \) is WSS with \( S_{\sim\sim}(\omega) = Q(\omega) / \pi \).

\[ \text{11-14} \quad E(\sim(t)) = 0 \quad E(\sim(\tau)) = \int_{-T}^{T} f(t)e^{-j\omega t} \, dt \]

The above integral is the transform of the product \( f(t)p_T(t) \), hence (frequency convolution theorem), it equals \( F(\omega)*\sin T\omega / \pi \omega \).

\[ \text{Var} \sim(\tau) = E \left\{ \left| \int_{-T}^{T} \sim(t)e^{-j\omega t} \, dt \right|^2 \right\} \]

The integral is the transform of the nonstationary white noise \( \sim(t)p_T(t) \). The autocorrelation of this process equals \( q(t_1)\delta(t_1 - t_2) \) where \( q(t) = qp_T(t) \). Hence, [see (11-69)]

\[ \text{Var} \sim(\tau) = Q(0) = \int_{-T}^{T} q(t) \, dt = 2qT \]
14-1 It suffices to show that \[ H(\Lambda \cdot B | B_j) = H(\Lambda | B_j) \]

Since

\[ A_{i_k} B_j = \begin{cases} A_{i_k} B_j & k = j \\ \emptyset & k \neq j \end{cases} \]

and \( P(A_{i_k} B_j | B_j) = P(A_{i_k} | B_j) \)

(14-40) yields

\[
H(\Lambda \cdot B | B_j) = - \sum_{i, k} P(A_{i_k} B_j | B_j) \log P(A_{i_k} B_j | B_j)
\]

\[
= - \sum_i P(A_i | B_j) \log P(A_i | B_j) = H(\Lambda | B_j)
\]

14-2 If \( \alpha < \beta \), then \( \phi'(\alpha) > \phi'(\beta) \) because

\[
\phi'(\alpha) - \phi'(\beta) = \log(\beta/\alpha) > 0.
\]

Hence,

\[
\int_a^b \phi'(\alpha) d\alpha > \int_{a+c}^{b+c} \phi'(\alpha) d\alpha, \quad c > 0
\]

This yields

\[
\phi(p_1 + p_2) - \phi(p_1) = \int_{p_1}^{p_1 + p_2} \phi'(\alpha) d\alpha < \int_0^{p_2} \phi'(\alpha) d\alpha = \phi(p_2)
\]

Similarly

\[
\phi(p_1 + \epsilon) - \phi(p_1) - \phi(p_2) + \phi(p_2 - \epsilon)
\]

\[
= \int_{p_1}^{p_1 + \epsilon} \phi'(\alpha) d\alpha - \int_{p_2 - \epsilon}^{p_2} \phi'(\alpha) d\alpha > 0
\]
Applying the identity

\[ H(A_1 \cdot A_2) = H(A_1) + H(A_2 | A_1) \]  \hspace{1cm} (i)

to the partitions \( A_1 = A, A_2 = B \cdot C \) and \( A_1 = A \cdot B, A_2 = C \), we obtain the first line. The second line follows from the first [see (i)]. The third line is a consequence of the first two.

It follows if we apply the identity

\[ I(A_1, A_2) = H(A_1) + H(A_2) - H(A_1 \cdot A_2) \]

to the partitions \( A_1 = A \cdot B, A_2 = C \).
(a) From (14-53)

\[
I(A, B \cdot C) = H(A) + H(B \cdot C) - H(A \cdot B \cdot C)
\]

\[
I(A, C) = H(A) + H(C) - H(A \cdot C)
\]

and since (see Prob. 14-4)

\[
H(A \cdot B \cdot C) - H(A \cdot C) = H(A \cdot B | C) - H(A | C)
\]

we conclude with (14-49) that

\[
I(A, B \cdot C) - I(A \cdot C) = H(B | C) + H(A | C) - H(A \cdot B | C)
\]

(b) If B \cdot C is observed, then the resulting prediction in the uncertainty of A equals I(A, B \cdot C). But, if B \cdot C is observed, then C is observed, hence, the reduction in the uncertainty of A is at least I(A, C). Hence

\[
I(A, B \cdot C) \geq I(A, C)
\]

with equality only if I(A, B|C) = 0, i.e., if in the subsequence of trials in which C occurred, knowledge of the occurrence of B gives no information about A.
The partition $H(A^3)$ has eight elements with respective probabilities

$$p^3, p^2q, p^2q, p^2q, pq, pq, p^2q, q^2$$

Hence

$$H(A^3) = -p^3 \log p^3 - 3p^2q \log p^2q - 3pq^2 \log pq^2 - q^3 \log q^3$$

$$= -3p(p^2 + 2pq + q^2) \log p - 3q(p^2 + 2pq + q^2) \log q$$

$$= -3p \log p - 3q \log q = 3H(A)$$

The density of the RV $w = x + a$ equals $f_x(w-a)$. Hence,

$$H(x+a) = -\int_{-\infty}^{\infty} f_x(w-a) \log f_x(w-a) dw$$

$$= -\int_{-\infty}^{\infty} f_x(x) \log f_x(x) dx = H(x)$$

The joint density of the RVs $x$ and $z = x + y$ equals $f_{xy}(x,z-x)$. Hence [see (14-90)]

$$H(z|x) = -\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{xy}(x,z-x) \log f_{xy}(x,z-x) f_x(x) dx dz$$

$$= -\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{xy}(x,y) \log f_{xy}(x,y) f_x(x) dy dx = H(y|x)$$

The RVs $x$ and $y$ take the values $x_i$ and $y_j$ respectively if $z = x_i + y_j$ iff $x = x_i$ and $y = y_j$ (assumption). Hence,

$$\{z = x_i + y_j\} = \{x = x_i\} \cap \{y = y_j\}$$

This shows that $A_z = A_x \cdot B_y$. Furthermore, since the RVs $x$ and $y$ are independent, the events $\{x = x_i\}$ and $\{y = y_j\}$ are also independent. This shows that the partitions $A_x$ and $B_y$ are independent and [see (14-44) and Prob. 14-1]
\[ H(A_z | A_x) = H(A_x \cdot A_y | A_x) = H(A_y | A_x) = H(A_y) \]

From this it follows that \( H(z | x) = H(y) \) because [see (14-88) and (14-41)]

\[ H(z | x) = H(A_z | A_x) \]

14-9 As we see from (14-80)

\[ H(x) = \ln a \text{ where we assume that } a = N\delta. \text{ The RV } y \text{ takes the values } 0,\delta,\ldots,(N-1)\delta \text{ with probability } 1/N. \text{ The conditional density of } \gamma \text{ assuming } y = k\delta \text{ is uniform in the interval } (k\delta,k\delta+\delta). \text{ Hence,} \]

\[ H(x | y = k\delta) = -\int_{k\delta}^{k\delta+\delta} f(x | y = k\delta) \ln f(x | y = k\delta) dx = \ln \delta \]

And as in (14-41)

\[ H(x | y) = \sum_{k=0}^{N} H(x | y = k\delta) P(y = k\delta) = \ln \delta \]

Finally [see (14-95)]

\[ I(x, y) = H(x) - H(x | y) = \ln a - \ln \delta \]

14-10 If \( y_i = g(x_i), y_j = g(x_j) \) and \( y_i = y_j \) then \( x_i = x_j \). Hence,

\[ p_{i,j} = \begin{cases} p_i & i = j \\ 0 & i \neq j \end{cases} \]

\[ p_i = P(x = x_i) \]

and

\[ H(x, y) = -\sum_{i,j} p_{i,j} \log p_{i,j} = -\sum_{i} p_i \log p_i = H(g) \]
14-11 From Prob. 10-10 it follows with \( g(x) = x \) that \( H(x, x) = H(x) \).
And since \( [\text{see (14-103)}] \ H(x, x) = H(x|x) + H(x) \) we conclude that \( H(x|x) = 0 \). From Prob. 14-3 it follows that
\[
H(y, x|x) = H(A_y, A_x|A_y A_x) = H(A_y|A_x A_x) + H(A_y|A_y A_x)
= H(A_y|A_x) = H(y|x)
\]
because \( A_x A_x = A_x \) and \( H(A_x|A_x) = H(x|x) = 0 \).

14-12 \( E(x) = 0 \) \( E(x^2) = 5 \) \( E(y) = 0 \)
\[
E(y^2) = \sum_{k=0}^{\infty} 2^{-2k} E(x^2) = \frac{20}{3} \]
\( E(x, y) = E(x^2) = 5 \)

(a) From \( (14-95), (14-84), \) and \( (15-86) \) with \( \mu_{11} = 5, \mu_{22} = 20/3, \) and \( \mu_{12} = 5 \)
\[
H(x) = \ln 10 \pi e \quad H(y) = \ln 40 \pi e / 3 \quad H(x, y) = \ln (10 \pi e / \sqrt{3})
\]
\( I(x, y) = \ln 2 \)

(b) The process \( y(t) \) is the output of the system
\[
L(z) = \frac{1}{1 - 0.5 z^{-1}} \quad z_0 = 1
\]
with input \( x_n \). Since \( \bar{H}(x) = H(x) \) and \( [\text{see (12A-1)}] \)
\[
\frac{1}{2\pi} \int_{-\pi}^{\pi} \ln |L(e^{j\phi})| d\phi = \ln z_0 = 0
\]
(14-133) yields \( \bar{H}(y) = \bar{H}(x) = H(x) = \ln 10 \pi e \).
14-13

\[ \bar{H}(x) = H(x) = -\frac{1}{2} \int_{4}^{6} \ln \frac{1}{2} \, dx = \ln 2 \]

And as in Prob. 14-12 with \( t_0 = 5 \),

\[ \bar{H}(y) = \bar{H}(x) + \ln 5 = \ln 10 \]

14-14 Given that \( f(x) = 0 \) for \(|x| > 1 \) and \( E(x) = 0.3 \), find \( f(x) \). With \( g(x) = x \), (14-143) yields

\[ f(x) = Ae^{-\lambda x} \]

where

\[ A \int_{-1}^{1} e^{-\lambda x} \, dx = \frac{A}{\lambda} (e^\lambda - e^{-\lambda}) = 1 \]

\[ A \int_{-1}^{1} xe^{-\lambda x} \, dx = \frac{A}{\lambda^2} (e^\lambda - e^{-\lambda}) - \frac{A}{\lambda} (e^\lambda - e^{-\lambda}) = 0.31 \]

Solving, we obtain \( A \approx 0.425, \lambda \approx -1 \)

14-15 \( f(x) = Ae^{-\lambda x} \) for \( 1 < x < 5 \) and 0 otherwise,

\[ A \int_{1}^{5} e^{-\lambda x} \, dx = 0.31 \quad A \int_{1}^{5} xe^{-\lambda x} \, dx = 3 \frac{37}{60} \]

Hence, \( A \approx 1.06, \lambda \approx 0.5 \)
14-16 From (14-151) with \( x_k = k \), \( g_1(x_k) = g_1(k) = k, k=1, \ldots, 6 \)

\[
g_2(x_k) = \begin{cases} 
0 & k=1,3,5 \\
1 & k=2,4,6 
\end{cases}
\]

\[
p_k = \begin{cases} 
A e^{-\lambda_1 k} & k=1,3,5 \\
A e^{-\lambda_2} & k=2,4,6 
\end{cases}
\]

Since \( p_1 + p_3 + p_6 = 0.5 \) and \( \bar{E}(\bar{x}) = 4.44 \), we conclude with \( z = e^{-\lambda_2} \) and \( w = e^{-\lambda_2} \) that

\[
A(z+z^3+z^5) = Aw(z^2+z^4+z^6)
\]

\[
\sim A(z+3z^3+5z^5) + Aw(2z^2+4z^4+6z^6) = 4.44
\]

This yields \( A \approx 0.0437 \), \( z = 1/w \approx 1.468 \)

14-17 (a) The transformation \( y = 3z \) is one-to-one, hence, \( H(y) = H(z) \)

(b) From (14-113) with \( g(x) = 3x \): \( H(y) = H(z) + \ln 3 \)

14-18 (a) For fair dice, \( P(7) = \frac{1}{6} \), \( P(11) = \frac{1}{18} \), \( P(\text{neither 7 nor 11}) = \frac{14}{18} \)

\[
H(A) = - \left( \frac{1}{6} \ln \frac{1}{6} + \frac{1}{18} \ln \frac{1}{18} + \frac{14}{18} \ln \frac{14}{18} \right) = 0.655
\]

(b) From (14-10) with \( n=100 \) and \( N=3 \):

\[
n_T \approx e^{nH(A)} \approx 2.79 \times 10^{28} \quad n_a \approx N^n \approx 5.16 \times 10^{47}
\]
The process $x_n$ is WSS with entropy rate $\bar{H}(x)$. Show that, if

$$w_n = \sum_{k=0}^{n} x_{n-k} \ell_k$$

then

$$\lim_{n \to \infty} \frac{1}{n+1} H(w_0, \ldots, w_n) = \bar{H}(x) + \ln |\ell_0|$$  \hspace{1cm} (i)

**Proof.** The RVs $w_0, \ldots, w_n$ are linear transformations of the RVs $x_0, \ldots, x_n$ and the transformation matrix equals

$$
\begin{bmatrix}
\ell_0 & 0 & \cdots & 0 \\
\ell_1 & \ell_0 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots \\
\ell_n & \ell_{n-1} & \cdots & 0 \\
\end{bmatrix}
$$

Since the determinant of this transformation equals $|\ell_0|^{n+1}$, (14-115) yields

$$H(w_0, \ldots, w_n) = H(x_0, \ldots, x_n) + (n+1)\ln |\ell_0|$$

Dividing by $(n+1)$ we obtain (i) as $n \to \infty$.

---

As in Example 14-19, $f(p) = A e^{-\lambda p}$. To find $\lambda$, we use the $\lambda$-$\eta$ curve of Fig. 14-16. This yields

$$\lambda = -1.23 \quad f(p) = 0.51 e^{1.23p}$$
14-21 As in Example 14-22, \( p_k = A e^{-\lambda k} \). To find \( \lambda \), we use the \( v-n \) curve of Fig. 14-17. This yields (see also Jaynes)

\[
\begin{align*}
\lambda &= -0.371 \\
w &= 1.449
\end{align*}
\]

<table>
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<tr>
<th>( p_1 )</th>
<th>( p_2 )</th>
<th>( p_3 )</th>
<th>( p_4 )</th>
<th>( p_5 )</th>
<th>( p_6 )</th>
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<td>0.054</td>
<td>0.079</td>
<td>0.114</td>
<td>0.165</td>
<td>0.240</td>
<td>0.348</td>
</tr>
</tbody>
</table>

14-22 The unknown density is normal as in (14-157) where

\[
\Delta = \begin{vmatrix}
4 & 1 & 1 \\
1 & 4 & m_{23} \\
1 & m_{23} & 4
\end{vmatrix} = -4m_{23}^2 + 2m_{23} + 56
\]

The moment \( m_{23} = E(x_2x_3) \) must be such as to maximize \( \Delta \). This yields \( m_{23} = 0.25 \).

<table>
<thead>
<tr>
<th>14-23</th>
<th>( p_i )</th>
<th>0.3</th>
<th>0.2</th>
<th>0.15</th>
<th>0.15</th>
<th>0.1</th>
<th>0.06</th>
<th>0.04</th>
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<tbody>
<tr>
<td>( \sum_{i=1}^{L} \frac{1}{2} \mathbf{m}_i )</td>
<td>( \frac{1}{4} \leq p_i &lt; \frac{1}{2} )</td>
<td>( \frac{1}{8} \leq p_i &lt; \frac{1}{4} )</td>
<td>( \frac{1}{16} \leq p_i &lt; \frac{1}{8} )</td>
<td>( \frac{1}{32} \leq p_i &lt; \frac{1}{16} )</td>
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<td>Shannon</td>
<td>( n_i )</td>
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<td>3</td>
<td>3</td>
<td>4</td>
<td>5</td>
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<td>1</td>
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<tr>
<td>( x_i )</td>
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<td>010</td>
<td>011</td>
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<td>101</td>
<td>110</td>
<td>111</td>
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\( L = 2.7 \)
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<tbody>
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<tr>
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<tr>
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<td>$A_{10}$</td>
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<td>$A_{11}$</td>
<td>0.2</td>
<td>$A_{110}$</td>
<td>0.1</td>
</tr>
<tr>
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<td>$A_{100}$</td>
<td>0.15</td>
<td>$A_{101}$</td>
<td>0.15</td>
<td>$A_{110}$</td>
<td>0.4</td>
</tr>
<tr>
<td>$X_i$</td>
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<td>01</td>
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</table>

$L = 2.6$

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<table>
<thead>
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<tbody>
<tr>
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<td>01</td>
<td>10</td>
<td>11</td>
<td>10</td>
<td>11</td>
</tr>
</tbody>
</table>

$L = 2.6$
If $x_n = 0$, then $\bar{x}_n = 000$ and $y_n = 1$ iff $\bar{y}_n$ consists of one 0 or no zeros. The probability of one and only one zero equals $3\beta^2(1-\beta)$ [see (3-13)]; the probability of no zeros equals $\beta^3$. Hence,

$$P(y_n = 1 | x_n = 0) = 3\beta^2(1-\beta) + \beta^3$$

Thus, the redundantly coded channel of Example 14-29 is symmetrical as in (14-191) with probability of error $\beta_1 = \beta^2$.

If the received information is always wrong, then

$$P(y_n = 1 | x_n = 0) = \beta = 1$$

hence $C = 1 - r(\beta) = 1$.